FDPE Microeconomics 3
Spring 2017
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## Hints for Problem Set 1

1. Consider strategic form games with two players and two actions available for each player. Let $\Sigma_i$ denote the set of mixed strategies for each player. Denote by $B_i(\sigma_j) \subset \Sigma_i$ the set of best responses of player $i$ to player j's strategy $\sigma_j$. In other words,

$$B_i(\sigma_j) := \{\sigma_i \in \Sigma_i | u_i(\sigma_i, \sigma_j) \geq u_i(\sigma_i^{'}, \sigma_j) \text{ for all } \sigma_i^{'} \in \Sigma_i\}$$

For each of the classic 2x2 games listed in the lecture notes (prisoner's dilemma, stag hunt, battle of sexes, hawk-dove, matching pennies) derive $B_1(\sigma_2)$ for all $\sigma_2 \in \Sigma_2$. Draw a figure of the best-response correspondence in each case. Find all Nash equilibria of the games.

**Solutions**

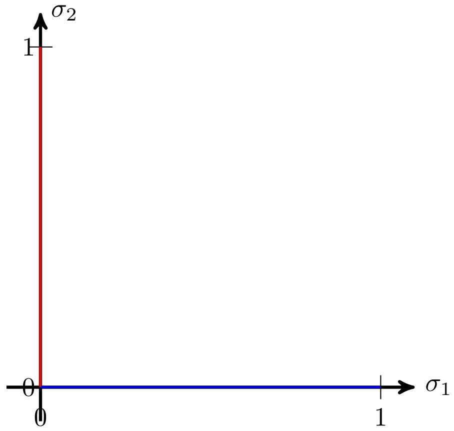
For all best-response figures, I denote the probabilities that player 1 takes the action on the first row by $\sigma_1$ and that player 2 takes the action on the first column by $\sigma_2$. Red and blue curves denote player 1's and player 2's best-responses respectively.

**Prisoner's dilemma**

| | C | D |
|---|---|---|
| C | $3,3$ | $0,4$ |
| D | $4,0$ | $1,1$ |

It is easy to see from the payoff matrix that defecting (D) strictly dominates cooperation (C). Therefore, the best response is always to defect, i.e. $B_1(\sigma_2) = 0$ with $\sigma_i$ denoting the probability that player $i$ plays C. Hence, $(D, D)$ is the only Nash equilibrium of the game.

Best-response curves:




**Stag hunt**

| | A | B |
|---|---|---|
| A | $2,2$ | $0,1$ |
| B | $1,0$ | $1,1$ |

To derive the best response correspondence here we need to do a little bit more work than in the previous case. Let's denote the probability that player $i$ plays A by $\sigma_i$. Then the payoff from playing A yields player 1 a payoff:

$$\sigma_2 \cdot 2 + (1 - \sigma_2) \cdot 0$$

and playing B yields

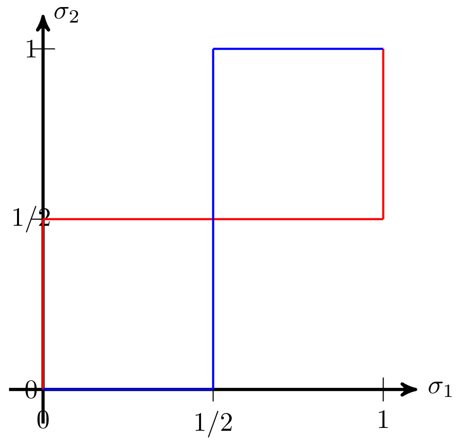
$$\sigma_2 \cdot 1 + (1 - \sigma_2) \cdot 1$$

These are equal when $\sigma_2 = \frac{1}{2}$. The best response correspondence thus is

$$B_1(\sigma_2) = \begin{cases} 1 \text{ if } \sigma_2 \geq \frac{1}{2} \\ 0 \text{ if } \sigma_2 \leq \frac{1}{2} \\ [0,1] \text{ if } \sigma_2 = \frac{1}{2} \end{cases}$$

There are three Nash equilibria in this game: (A,A), (B,B) and $(\frac{1}{2}, \frac{1}{2})$

Best-response curves:




**Battle of the sexes**

| | B | F |
|---|---|---|
| B | $2,1$ | $0,0$ |
| F | $0,0$ | $1,2$ |

Let $\sigma_i$ denote the probability that player $i$ plays B. Player 1 is indifferent between ballet (B) and football (F) when:

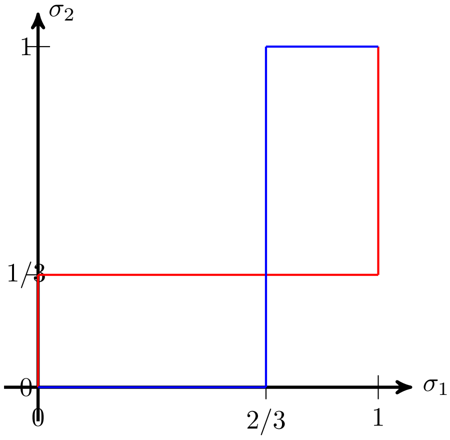
$$\sigma_2 \cdot 2 + (1 - \sigma_2) \cdot 0 = \sigma_2 \cdot 2 + (1 - \sigma_2) \cdot 1$$

Solving this yields $\sigma_2 = \frac{1}{3}$. The best response correspondence is then

$$B_1(\sigma_2) = \begin{cases} 1 \text{ if } \sigma_2 \geq \frac{1}{3} \\ 0 \text{ if } \sigma_2 \leq \frac{1}{3} \\ [0,1] \text{ if } \sigma_2 = \frac{1}{3} \end{cases}$$

There are three Nash equilibria: (B,B), (F,F), $(\frac{2}{3}, \frac{1}{3})$.

Best-response curves:




**Hawk-Dove**

| | D | H |
|---|---|---|
| D | $3,3$ | $1,4$ |
| H | $4,1$ | $0,0$ |

Let $\sigma_i$ denote the probability that player $i$ plays D. Player 1 is indifferent between being dovish (D) and hawkish (H), when

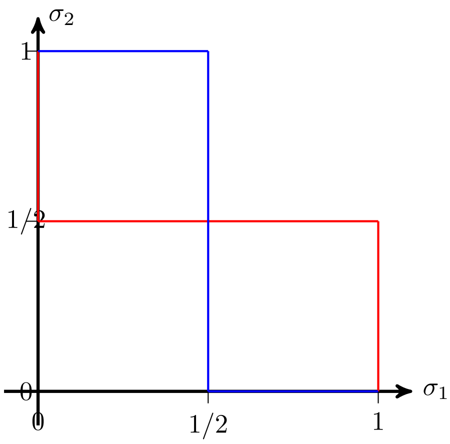
$$\sigma_2 \cdot 3 + (1 - \sigma_2) \cdot 1 = \sigma_2 \cdot 4 + (1 - \sigma_2) \cdot 0$$

Solving this yields $\sigma_2 = \frac{1}{2}$. The best response correspondence is thus

$$B_1(\sigma_2) = \begin{cases} 1 \text{ if } \sigma_2 \leq \frac{1}{2} \\ 0 \text{ if } \sigma_2 \geq \frac{1}{2} \\ [0,1] \text{ if } \sigma_2 = \frac{1}{2} \end{cases}$$

There are again three Nash equilibria: (H,D), (D,H) and $(\frac{1}{2}, \frac{1}{2})$.

Best-response curves:




**Matching pennies**

| | H | T |
|---|---|---|
| H | $1,-1$ | $-1,1$ |
| T | $-1,1$ | $1,-1$ |

Letting $\sigma_i$ denote now that the probability that player $i$ plays H, then the indifference condition is:

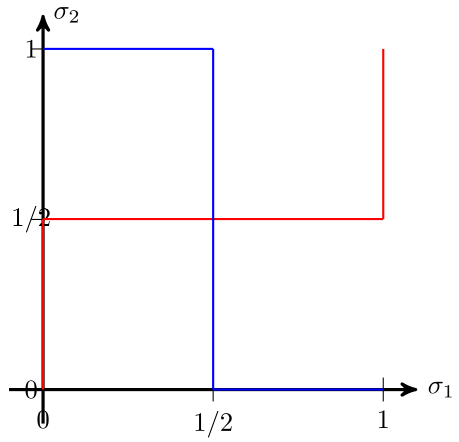
$$\sigma_2 \cdot 1 + (1-\sigma_2)\cdot(-1) = \sigma_2 \cdot (-1) + (1-\sigma_2)\cdot 1$$

Solving this yields $\sigma_2 = \frac{1}{2}$. The best response correspondence is

$$B_1(\sigma_2) = \begin{cases} 1 \text{ if } \sigma_2 \geq \frac{1}{2} \\ 0 \text{ if } \sigma_2 \leq \frac{1}{2} \\ [0,1] \text{ if } \sigma_2 = \frac{1}{2} \end{cases}$$

There is only one Nash equilibrium: $(\frac{1}{2}, \frac{1}{2})$

Best-response curves:




2. (Guess the average). Consider the n-player game where all the players announce simultaneously a number in the set $\{1,\dots,K\}$ and a price of \$1 is split equally among all the players having the guess closest to 2/3 of the average of the announced numbers. Find the strategies that are rationalizable (i.e. survive iterated elimination of strictly dominated strategies) and all Nash equilibria of the game.

   **Solution.**

   None of the strategies is strictly dominated by a pure strategy. However, unreasonable strategies to guess more than $\frac{2}{3}K$ are strictly dominated by some mixed strategies since they are never-best responses to any belief. After these strategies are ruled out, strategies to guess more than $\left(\frac{2}{3}\right)^2 K$ becomes unreasonable. By continuing this way, one will be left with one rationalizable strategy only: there is a unique NE where everybody plays 1. Below, I present a systematic iteration process that deletes the highest remaining action in each round.

   Let's first fix some notation. $s_{-i}$ denotes a pure strategy profile for everybody else except player $i$. The set of all pure strategies of player $i$ is $\{1,\dots,K\}$.

   In order to the strategy $K$ to be strictly dominated by a (probably mixed) strategy $\sigma_i$, we have to have:

$$\begin{aligned} & u_i(\sigma_i, s_{-i}) > u_i(K, s_{-i}) \quad \forall s_{-i} \\ \overset{\text{here}}{\Leftrightarrow} & \begin{cases} u_i(\sigma_i, (K,\dots,K)_{-i}) > u_i(K,(K,\dots,K)_{-i}) \\ u_i(\sigma_i, s_{-i}) > 0 \quad \forall s_{-i} \neq (K,\dots,K)_{-i}. \end{cases} \end{aligned}$$

   Both inequalities are satisfied (check this!) by a strategy that puts a weight 2/3 on an action $K-1$ and splits remaining 1/3 equally among all smaller actions, giving weight of $1/3\frac{1}{K-2}$ on each.

   For the next iteration round, use similar mixed strategy to delete $K-1$. Proceed this way and delete all actions above 1.

   The mixed strategy used here is only one example of strategies dominating the highest action. In addition, one could delete more than one pure strategy of one player in each round, but then different dominating mixed action should be used.

3. There are two players with strategy spaces $S_i = \{1, 2, 3\}$, $i = 1, 2$. Each player wants to choose the highest possible number, but both get zero if they pick the same number:

$$u_i(s_i, s_j) = \begin{cases} s_i & \text{if } s_i \neq s_j \\ 0 & \text{if } s_i = s_j \end{cases}, \quad i, j = 1, 2, i \neq j.$$

(a) Which strategies are rationalizable? (i.e. survive iterated elimination of strictly dominated strategies)

**Solution.** Let us write the game in matrix form:

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $0,0$ | $1,2$ | $1,3$ |
| 2 | $2,1$ | $0,0$ | $2,3$ |
| 3 | $3,1$ | $3,2$ | $0,0$ |

It is easy to see that strategy 1 is never-best response to a pure strategy and although we cannot see it directly from the payoff matrix, it turns out that it is not a best-response to any belief (distribution over other player's strategies) that a player may hold. We will show this by using the the fact that the set of rationalizable strategies and the set of strategies that survive the iterated domination of stricly dominated strategies coincide in games with two players. We thus need to find a mixed strategy that involves mixing between 2 and 3, which strictly dominates 1. Let's denote this strategy by $\sigma_i = (0, \sigma_i, 1 - \sigma_i)$. Note that

$$u_i(\sigma_{\mathbf{1}}, 1) = 2\sigma_i + 3(1 - \sigma_i) \quad > \quad 0 = u_i(1, 1), \qquad \forall\, \sigma_i \in [0, 1];$$

$$u_i(\sigma_{\mathbf{1}}, 2) = 3(1 - \sigma_i) \quad > \quad 1 = u_i(1, 2), \qquad \forall\, \sigma_i < \tfrac{2}{3};$$

$$u_i(\sigma_{\mathbf{1}}, 3) = 2\sigma_i \quad > \quad 1 = u_i(1, 3) \qquad \forall\, \sigma_i > \tfrac{1}{2}.$$

Therefore, any strategy $\sigma_{\mathbf{i}} = (0, \sigma_i, 1 - \sigma_i)$ such that $\sigma_i \in \left(\frac{1}{2}, \frac{2}{3}\right)$ strictly dominates strategy $s_i = 1$ for both $i = 1, 2$. Thus the set of rationalizable strategies are $S_i = \{2, 3\}$. We know that 2 and 3 are in this set already from looking at the pure strategies.

(b) Find all (pure and mixed) Nash equilibria of the above game.

**Solution.** Since all Nash equilibria involve playing best responses, we can restrict ourselves to think about strategies $\{2, 3\}$. There are two equilibria in pure strategies, (2,3) and (3,2), and one in mixed strategies, where the players mix between 2 and 3 with probabilities $((\frac{2}{5}, \frac{3}{5}))$. The latter can be found again from the indifference condition between the two strategies ($\sigma_2$ denotes the probability that player 2 plays 2):

$$(1 - \sigma_2)2 = \sigma_2 3 \Leftrightarrow \sigma_2 = \frac{2}{5}$$

(c) If you have time, think how you can generalize your answers to a game with strategy spaces $S_i = \{1, ..., N\}$.

**Solution.** When the strategy space is $S_i = \{1, ..., N\}$, the two pure strategy Nash equilibria of the game are $(N - 1, N)$ and $(N, N - 1)$. This is easy to see from the payoff structure. Generally, the mixed strategies are more complicated than in the above case, since they might involve mixing with more than two pure strategies. Thus also the set of rationalizable strategies is larger than just the top two integers. The set of rationalizable strategies doesn't generalize in the sense that it contains more than just the two highest integers if $N > 2$. However, it is always true that only some $M(N)$ highest integers are rationalizable. Intuitively, one can think that as $N$ grows, the more the players stand to lose by choosing the same number and also the "insurance" of choosing a lower number costs less. Thus in a mixed strategy, one wants to mix between more than two pure strategies.

4. Consider the following two-player game:

| | L | R |
|---|---|---|
| U | $5, 1$ | $0, 0$ |
| D | $4, 4$ | $1, 5$ |

(a) Find all Nash equilibria of the game. What is the best payoff that the players can get in a symmetric equilibrium?

**Solution.** There are three Nash equilibria in this game: (U,L),(D,R) and $(\frac{1}{2}, \frac{1}{2})$. The best symmetric equilibrium is the mixed strategy equilibrium, since the pure strategy equilibria are not symmetric. It yields an expected payoff of $\frac{5+4+1}{4} = 2.5$.

(b) Suppose that before choosing their actions, the players first toss a coin. After publicly observing the outcome of the coin toss, they choose simultaneously their action. Draw the extensive form game and define available strategies for the players. Find a Nash equilibrium that gives both players a higher payoff than the symmetric equilibrium in a).

**Solution.** Players can now condition their actions on the coin toss. We can take this into account by for example denoting the strategies by double letters (UU), where the first one stands for the action after heads and the latter one after tails. The set of strategies are thus $S_1 = \{UU, UD, DU, DD\}$ and $S_2 = \{LL, LR, RL, RR\}$. The coin toss allows the players to avoid the "mistake" of ending in (U,R) in mixed strategies: a Nash equilibrium is, for example, when player 1 plays U on heads and D on tails and player 2 plays L on heads and R on tails. There are no profitable deviations, because the players are playing the two pure strategy Nash equilibria with probability 1/2 each. This yields an expected payoff of $\frac{5+1}{2} = 3$ and thus is better than the mixed strategy equilibrium in (a) for both players.

The extensive form will be drawn in class.

(c) Suppose that there is a mediator that can make a recommendation separately for each player. Suppose that the mediator makes recommendation (U,L), (D,L) or (D,R), each with probability 1/3. Each player only observes her own action choice recommendation (so that, e.g. player one upon seeing recommendation D does not know whether the recommended profile is (D,L) or (D,R)). Does any of the players have an incentive to deviate from the recommended action? What is the expected payoff under this scheme?

**Solution.** Let's go through the recommended strategies case by case:

Player 1

Recommendation: U $\Rightarrow$ player 1 knows that player 2 will play L. No incentive to deviate, since the payoff from U is higher than from D $(5 > 4)$.
Recommendation: D $\Rightarrow$ player 2 will play L or R with equal probability. Player 1 is indifferent between U and D: $\frac{1}{2} \cdot 5 + 0 \cdot \frac{1}{2} = 4 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$. So no incentive to deviate.

Player 2

Recommendation: L $\Rightarrow$ player 2 knows that player 1 will play U and D with equal probability. Player 2 is indifferent between L and R: $\frac{1}{2} \cdot 4 + 1 \cdot \frac{1}{2} = 5 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}$.
Recommendation: R $\Rightarrow$ player 2 knows that player 1 will play D. Payoff from R strictly higher to L $(5 > 4)$.

The players, therefore, have no incentive to deviate. The expected payoff from the scheme is $\frac{1+4+5}{3} = 3\frac{1}{3}$. This is because (D,L), which offers the highest summed payoff, is now played with probability $\frac{1}{3}$.

5. Consider a simple model of R&D race. Two firms choose simultaneously how much money to invest and the winner is the firm who invests more. The winner gets a prize worth 1 million Euros. If both firms invest the same amount, then each firm wins with probability 1/2. The loser gets no prize. Formulate this situation as a strategic form game and analyze it (hint: look for a symmetric Nash equilibrium in mixed strategies i.e. a probability density function for investment level that keeps players indifferent between different amounts).

**Solution.** Let's set the unit to million Euros. The pure strategy space for firm $i$ is $S_i := [0, \infty)$. The payoff function of firm $i$ is given by

$$u_i(s_1, s_2) = \begin{cases} -s_i & \text{if } s_i < s_j, \\ \frac{1}{2} - s_i & \text{if } s_i = s_j, \\ 1 - s_i & \text{if } s_i > s_j. \end{cases}$$

where $j \neq i$.

It is easy to see that investing more than 1 is strictly dominated by investing 0 and these strategies can be removed. We look for a symmetric Nash equilibrium in mixed strategies where the firms' strategy has no atom, i.e., it can be represented by a probability density function $f$. It is easy to see that the support set of the probability density function $f$ must contain 0, otherwise, one of the firms can be made better off by moving some of the probability mass around the lowest investment level in the support set to 0. Similarly, the support set must be a connected set. (Why?) Let $[0, \overline{s}]$ be the support set. Using the indifferent condition, we have, for all $s \in [0, \overline{s}]$,

$$F(s) - s = \int_0^s f(x)\, dx - s = 0.$$

Thus,

$$F(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1, \\ 1 & \text{if } s > 1. \end{cases}$$

The symmetric Nash equilibrium involves mixing uniformly between 0 and 1.

We can actually show that the Nash equilibrium in mixed strategies that we found is the unique Nash equilibrium by modifying the previous arguments and making use of the following result. (Show it!)

**Lemma.** In any Nash equilibrium in mixed strategies, at most one firm's strategy involves atoms and the atom can only be located at 0.

**Proof.** Suppose firm $i$'s equilibrium strategy involves an atom at some $x \in [0, 1]$, we would like to show that there exists some $\varepsilon > 0$ such that the probability that firm $j$ invests in $(x - \varepsilon, x]$ is 0 in equilibrium. This would imply that $x = 0$, as, otherwise, firm $i$ can move the probability mass at $x$ to a point in $(x - \varepsilon, x)$ and gets strictly better off. When $x = 0$, this also implies that firm $j$'s equilibrium strategy does not involve an atom at 0.

To see the result, suppose $x \in [0, 1)$, then, for $\varepsilon > 0$ small enough, firm $j$ can move the probability mass in $(x - \varepsilon, x]$ to a level just above $x$ and gets strictly better off. Suppose $x = 1$, then, for $\varepsilon > 0$ small enough, firm $j$ gets strictly negative payoff by investing in $(x - \varepsilon, x]$. As a result, firm $j$ will be better off by investing 0 instead.