

Hints for Problem Set 2

1. Consider a zero-sum game, where the sets of pure strategies are $S_i = \{1, \dots, K\}$, $i = 1, 2$, and payoffs are

$$u_1(s_1, s_2) = -u_2(s_1, s_2) = \begin{cases} 1 & \text{if } s_1 = s_2, \\ -1 & \text{if } s_1 \neq s_2. \end{cases}$$

- (a) Compute maxmin payoffs when only pure strategies are allowed:

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2) \quad \text{and} \quad \max_{s_2 \in S_2} \min_{s_1 \in S_1} u_2(s_1, s_2).$$

Solution.

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2) = -1,$$

because regardless of the number player 1 chooses, player 2 always mismatches and whenever this happens, the payoff of player 1 is -1 .

Analogously,

$$\max_{s_2 \in S_2} \min_{s_1 \in S_1} u_2(s_1, s_2) = -1,$$

because regardless of the number player 2 chooses, player 1 always matches and whenever this happens, the payoff of player 2 is -1 .

- (b) Compute

$$\min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2) \quad \text{and} \quad \min_{s_1 \in S_1} \max_{s_2 \in S_2} u_2(s_1, s_2).$$

Solution.

$$\min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2) = 1,$$

because regardless of the number player 2 chooses, player 1 always matches and whenever this happens, the payoff of player 1 is 1 .

And similarly

$$\min_{s_1 \in S_1} \max_{s_2 \in S_2} u_2(s_1, s_2) = 1,$$

because regardless of the number player 1 chooses, player 2 always mismatches and whenever this happens, the payoff of player 2 is 1 .

Notice that parts a) and b) combined tell us that

$$\begin{aligned} \max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2) &\neq \min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2), \\ \text{and} \quad \max_{s_2 \in S_2} \min_{s_1 \in S_1} u_2(s_1, s_2) &\neq \min_{s_1 \in S_1} \max_{s_2 \in S_2} u_2(s_1, s_2). \end{aligned}$$

- (c) Now, allow mixed strategies and compute

$$\max_{\sigma_1 \in \Delta(S_1)} \min_{\sigma_2 \in \Delta(S_2)} u_1(\sigma_1, \sigma_2) \quad \text{and} \quad \max_{\sigma_2 \in \Delta(S_2)} \min_{\sigma_1 \in \Delta(S_1)} u_2(\sigma_1, \sigma_2).$$

Solution.

Notice that given $\sigma_1 \in \Delta(S_1)$, the best response is given by

$$\sigma_2(s) = \begin{cases} 1, & \text{if } s \in \{s' \in S_1 \mid \sigma_1(s') \leq \sigma_1(s'') \ \forall s'' \in S_1\} \\ 0, & \text{otherwise.} \end{cases}$$

That is, player 2 picks an integer which player 1 chooses with the smallest probability. Knowing this, the best-reply of player 1 is to distribute the probability mass evenly across all integers, and hence $\sigma_1(s) = \frac{1}{K}$ for all $s = 1, \dots, K$. Given these

$$\begin{aligned} \max_{\sigma_1 \in \Delta(S_1)} \min_{\sigma_2 \in \Delta(S_2)} u_1(\sigma_1, \sigma_2) &= \sum_{s \in S_1} \sigma_1(s) [\sigma_2(s) - (1 - \sigma_2(s))] \\ &= \frac{2-K}{K}. \end{aligned}$$

Analogously, for any strategy of player 2, $\sigma_2 \in \Delta(S_2)$, the best response is given by

$$\sigma_1(s) = \begin{cases} 1, & \text{if } s' \in \{s \in S_2 \mid \sigma_2(s') \geq \sigma_2(s'') \ \forall s'' \in S_2\} \\ 0, & \text{otherwise.} \end{cases}$$

That is, player 1 picks an integer on which player 2 puts the largest probability mass. Knowing this, the best-reply of player 2 is to distribute the probability mass evenly across all integers, $\sigma_2(s) = \frac{1}{K}$ for all $s = 1, \dots, K$, which yields

$$\begin{aligned} \max_{\sigma_2 \in \Delta(S_2)} \min_{\sigma_1 \in \Delta(S_1)} u_2(\sigma_1, \sigma_2) &= \sum_{s \in S_2} \sigma_2(s) [(1 - \sigma_1(s)) - \sigma_1(s)] \\ &= \frac{K-2}{K}. \end{aligned}$$

(d) Compute

$$\min_{\sigma_2 \in \Delta(S_2)} \max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2) \quad \text{and} \quad \min_{\sigma_1 \in \Delta(S_1)} \max_{\sigma_2 \in \Delta(S_2)} u_2(\sigma_1, \sigma_2).$$

Solution.

Given $\sigma_2 \in \Delta(S_2)$, the best response for player 1 is given by

$$\sigma_1(s) = \begin{cases} 1, & \text{if } s \in \{s' \in S_2 \mid \sigma_2(s') \geq \sigma_2(s'') \ \forall s'' \in S_2\} \\ 0, & \text{otherwise.} \end{cases}$$

That is, player 1 picks an integer which player 2 chooses with the the greatest probability. Knowing this, the best-reply of player 2 is to choose $\sigma_2(s) = \frac{1}{K}$ for all $s = 1, \dots, K$. Given these

$$\begin{aligned} \min_{\sigma_2 \in \Delta(S_2)} \max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2) &= \sum_{s \in S_1} \sigma_1(s) [\sigma_2(s) - (1 - \sigma_2(s))] \\ &= \frac{2-K}{K}. \end{aligned}$$

Analogously, for any strategy of player 1, $\sigma_1 \in \Delta(S_1)$, the best response of player 2 is

$$\sigma_2(s) = \begin{cases} 1, & \text{if } s \in \{s' \in S_1 \mid \sigma_1(s') \leq \sigma_1(s'') \ \forall s'' \in S_1\} \\ 0, & \text{otherwise.} \end{cases}$$

That is, player 2 picks an integer which player 1 chooses with the smallest probability. Knowing this, the best-reply of player 1 is $\sigma_1(s) = \frac{1}{K}$ for all $s = 1, \dots, K$. Given these

$$\begin{aligned} \min_{\sigma_1 \in \Delta(S_1)} \max_{\sigma_2 \in \Delta(S_2)} u_2(\sigma_1, \sigma_2) &= \sum_{s \in S_2} \sigma_2(s) [(1 - \sigma_1(s)) - \sigma_1(s)] \\ &= \frac{K-2}{K}. \end{aligned}$$

Notice that parts c) and d) combined tell us that

$$\begin{aligned} \max_{\sigma_1 \in \Delta(S_1)} \min_{\sigma_2 \in \Delta(S_2)} u_1(\sigma_1, \sigma_2) &= \min_{\sigma_2 \in \Delta(S_2)} \max_{\sigma_1 \in \Delta(S_1)} u_1(\sigma_1, \sigma_2), \\ \text{and} \quad \max_{\sigma_2 \in \Delta(S_2)} \min_{\sigma_1 \in \Delta(S_1)} u_2(\sigma_1, \sigma_2) &= \min_{\sigma_1 \in \Delta(S_1)} \max_{\sigma_2 \in \Delta(S_2)} u_2(\sigma_1, \sigma_2). \end{aligned}$$

(e) Find Nash equilibria in pure strategies.

Solution.

Since

$$\begin{aligned} \max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2) &\neq \min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2), \\ \text{and} \quad \max_{s_2 \in S_2} \min_{s_1 \in S_1} u_2(s_1, s_2) &\neq \min_{s_1 \in S_1} \max_{s_2 \in S_2} u_2(s_1, s_2), \end{aligned}$$

there are no Nash equilibria in pure strategies by the minmax theorem.

- (f) Find Nash equilibria in mixed strategies.

Solution.

Firstly, we know that a mixed strategy equilibrium always exists in a finite game. Secondly, by the minmax theorem, we know that the set of equilibrium strategies equals the set of strategies that yield the minmax payoffs. Hence, the set of Nash equilibria in mixed strategies equals

$$\left\{ (\sigma_1, \sigma_2) \mid \sigma_i(s) = \frac{1}{K}, \forall s \in S_i, i = 1, 2 \right\}.$$

2. (War of attrition) Two players are fighting for a prize whose current value at any time $t = 0, 1, 2, \dots$ is $v > 1$. Fighting costs 1 unit per period. The game ends as soon as one of the players stops fighting. If one player stops fighting in period t , he gets no prize and incurs no more costs, while his opponent wins the prize without incurring a fighting cost. If both players stop fighting at the same period, then neither of them gets the prize. The players discount their costs and payoffs with discount factor δ per period.

This is a multi-stage game with observed actions, where the action set for each player in period t is $A_i(t) = \{0, 1\}$, where 0 means continue fighting and 1 means stop. A pure strategy s_i is a mapping $s_i : \{0, 1, \dots\} \rightarrow A_i(t)$ such that $s_i(t)$ describes the action that a player takes in period t if no player has stopped the game in periods $0, \dots, t-1$. A behavior strategy $b_i(t)$ defines a probability of stopping in period t if no player has yet stopped.

- (a) Consider a strategy profile $s_1(t) = 1$ for all t and $s_2(t) = 0$ for all t . Is this a Nash equilibrium?

Solution. This is an equilibrium: given the behavior of player 2, player 1 has no incentive to fight. Player 2 gets utility v so he has no incentive to deviate.

- (b) Find a stationary symmetric Nash equilibrium, where both players stop with the same constant probability in each period.

Solution. (By stationary one means equilibria with strategies that are independent of t .) Let p be this probability of stopping. The condition for a mixed strategy equilibrium is that a player is indifferent between fighting and dropping out. In any period the utility from fighting in the present period is $pv + (1-p) \cdot (-1)$, since player 2 succumbs with probability p and fight with probability $1-p$. The continuation value (value of the future that arises after $(0,0)$) is zero. Players mix in the next period which implies that they are indifferent between fighting and stopping. Stopping gives a zero payoff, and hence the expected payoff after any action in the support of the mixed strategy is also zero. Therefore, we can ignore the continuation value. The utility from dropping out is 0. Thus the equilibrium condition is

$$pv + (1-p) \cdot (-1) = 0$$

$$p = \frac{1}{1+v}$$

- (c) Are the strategy profiles considered above subgame-perfect equilibria?

Solution. Yes, both in part a) and b). This is because all stationary Nash equilibria are subgame perfect equilibria for stationary multistage games. In the game in question, previous fights are sunk cost and the time horizon is infinite, and hence all periods are equivalent to the first period. Therefore, the same argumentation, which was used for period 1 in a) and b), can be used for later periods as well. All stationary NE satisfy the one-step deviation condition.

- (d) Can you think of other strategy profiles that would constitute a sub-game perfect equilibrium?

Solution. The equilibrium in (a) can obviously be reversed: where player 1 stops immediately and player 2 never stops: $s_1(t) = 0$ for all t and $s_2(t) = 1$ for all t . We could also combine profiles in a) and b). For example, the following is a SPE:

$$s_1 = (1, p, p, \dots)$$

$$s_2 = (0, p, p, \dots).$$

There is also a mixed strategy equilibrium, where players stop every second period with probability ρ , i.e. their strategies assign probabilities $(0, \rho, 0, \rho, 0, \dots)$ and $(\rho, 0, \rho, 0, \dots)$ to quitting. The

argument why this works is similar to the symmetric equilibrium. The important condition is that the player who is mixing between stopping and continuing must be indifferent (the value of the game is zero for her).

The player who is not mixing has a value: $\rho v + (1 - \rho)(-1)$. (The not mixing player will mix in the following period, and hence her continuation value is zero.) The player, who is mixing now, has a continuation value of $\delta(\rho v + (1 - \rho)(-1))$. Her indifference condition yields:

$$\begin{aligned}\delta(\rho v + (1 - \rho)(-1)) - 1 &= 0 \\ \Leftrightarrow \rho &= \frac{1 + \delta}{\delta(1 + v)}.\end{aligned}$$

Can you see why there cannot be a period in which both players fight with probability one?

3. Consider a two-player stopping game with a finite time horizon $t = 0, 1, 2, \dots, T$. At each period, both players choose simultaneously whether to stop or continue. The game ends as soon as one of the players stop. The payoffs are given by $u_1(t) = u_2(t) = t$, if the game ends at period t . If no player ever stops, both players get zero.

- (a) Find all Nash equilibria. Are there subgame perfect equilibria?

Solution. The strategy set for each player in any period t when the game is still on-going is given by $S_i(t) = \{C, S\}$, where C stands for “continue” and S for “stop”. As in the war of attrition, a pure strategy of player i is a function $s_i : \{0, 1, 2, \dots\} \rightarrow S_i(t)$ such that $s_i^t(t)$ describes the action that player i takes in period t if no player has stopped the game in periods $0, \dots, t - 1$.

What are the pure strategy Nash equilibria? Since the game ends as soon as either one of the players stop, all strategy profiles where both players stop simultaneously at some period $t \leq T$ are Nash equilibria; no player can gain by unilateral deviation. This type of Nash equilibrium strategy profiles (s_1, s_2) can be characterized by

$$s_i(t) = \begin{cases} C & , \text{if } t < t^* \\ S & , \text{if } t = t^* \end{cases}$$

where $i = 1, 2$, and $t^* \leq T$. Equilibrium strategies can be anything after t^* since the game never continues to these periods.

There are also two pure strategy Nash equilibria in which one of the player stops at T and the other one continues forever.

All outcomes of the Nash equilibria described above are subgame perfect equilibria outcomes as well, but now we have to assume equilibrium play for later periods also. In a SPE, the players choose S simultaneously only at a set of time $\{t_1^*, t_2^*, \dots\}$, where $t_1^*, t_2^*, \dots \leq T$ and at least one of them plays S in period T . In an actual play of the game, the game will end at t_1^* .

There is no nontrivial subgame perfect equilibrium in mixed strategies. A player, who mixes in period t , should be indifferent between stopping and continuing. Then one of the following would have to hold: a) the other player stops at t , b) the continuation value is equal to t . In an equilibrium, someone stops at the latest in T . Since payoffs are increasing in time, the continuation value cannot be t and b) is ruled out. It is best response to stop before T only if the other player stops for sure. Hence, one player stopping and the other one mixing is not an equilibrium before T . In period T , the only equilibrium condition is that at least one player chooses S . The other one can as well mix.

- (b) Let the time horizon be infinite, that is $t = 0, 1, \dots$. The same questions as in a)

Solution. The equilibrium strategies stay the same as in the previous case except for the equilibria where one player stops at T and the other one continues. However, if the players are playing strategies in which they both stop simultaneously, there are no profitable deviations. The difference between Nash equilibria and subgame perfect equilibria is again that in a NE it does not matter what happens after first period where players choose S . Note that both players continuing forever is not a Nash equilibrium since both players would gain by stopping in finite time.

- (c) The game is otherwise as in (b), but at every period where both players choose “continue” the game ends with exogenous probability $p > 0$. If that happens before any of the players chooses to stop then both players get zero. Find all Nash equilibria and subgame perfect equilibria of the game.

Solution. Let’s consider first the set of Nash equilibria. It is intuitively clear from a) that mixing cannot occur on the equilibrium path. Therefore, we can analyze the game by considering the first time each player chooses to stop. Let s_i be the first time period that player i chooses to stop.

Now there will be a period, T^* , after which the players will stop regardless of the other players’ strategy, if it is on the equilibrium path. To solve for T^* , suppose time period t has been reached. If player i stops now he gets t . If he continues and stops in the next period he gets $(1 - p)(t + 1)$. Continuing is weakly optimal as long as

$$(1 - p)(t + 1) \geq t \Leftrightarrow t \leq \frac{1 - p}{p}.$$

Thus, T^* is an integer satisfying $\frac{1-p}{p} \leq T^* < \frac{1-p}{p} + 1$. If the first inequality holds as an equality, players are indifferent between stopping at T^* and $T^* + 1$. We ignore this possibility in the following.

Thus, the set of Nash equilibria contains strategies that satisfy either i) $s_1 = s_2 \leq T^*$ or ii) $s_i = T^*, s_j \geq T^*$. Note that this result resembles the one in part a). Similarly, in a SPE, the players choose C simultaneously only at a set of time $\{t_1^*, t_2^*, \dots\}$, where $t_1^*, t_2^*, \dots \leq T^*$ and at least one of them plays S in period T^* .

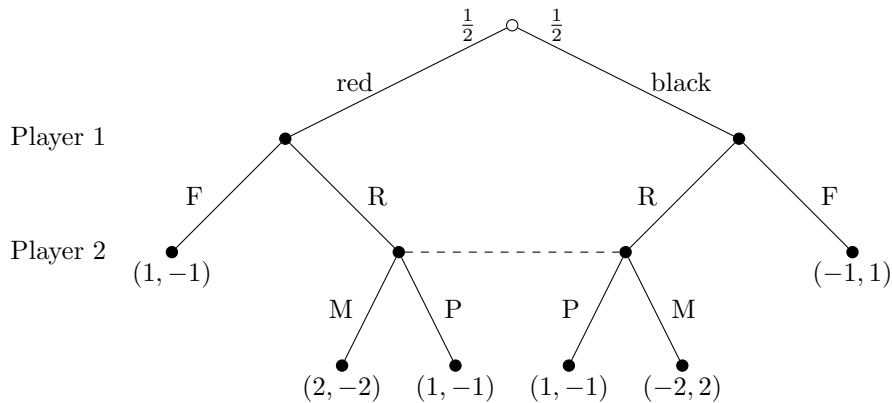
The normal form game is not conceptually the same as the original game, but it contains all of its strategic dimensions. Since only one history (nobody stopped and the game didn’t end endogenously) leads to new decision nodes, we don’t have to worry about history-dependent strategies, which would make dynamic game strategically different from a static one. One could, naturally, analyze part c) by using the original formulation in a similar manner as in parts a) and b).

4. Consider the simple card game discussed in the lecture notes: Players 1 and 2 put one dollar each in a pot. Then, player 1 draws a card from a stack, observes privately the card and decides whether to “raise” or “fold”. In case of “fold”, the game ends and player 1 gets the money if the card is red, while player 2 gets the money if black. In case of “raise”, player 1 adds another dollar in the pot, and player 2 must decide whether to “meet” or “pass”. In case of “pass”, game ends and player 1 takes the money in the pot. In case of “meet”, player 2 adds another dollar in the pot, and player 1 shows the card. Player 1 takes the money if the card is red, while player 2 takes the money if black.

- (a) Formulate the game as an extensive form game.

Solution. An extensive form game is defined by specifying:

- The set of players: $\mathcal{I} = \{1, 2\}$
- The order of moves, specified by the game tree, T .
- The players’ payoffs as a function of moves at the terminal nodes of the game tree.
- The players’ information sets at each node: $h \in H$.
- The available actions, when the players move: $A(h)$.
- Probability distribution over Nature’s moves: $P(\text{red}) = 0.5 = P(\text{black})$.



- (b) Represent the game in a strategic form and find the unique mixed strategy Nash equilibrium of the game.

Solution. We can use two payoff matrices to describe the game:

	M	P		M	P
R	2, -2	1, -1	R	-2, 2	1, -1
F	1, -1	1, -1	F	-1, 1	-1, 1
	The card is red			The card is black	

The strategy sets for the players are $S_1 = \{RR, RF, FR, FF\}$ and $S_2 = \{M, P\}$, i.e. player 1 can condition her action on the color of the card. To find the Nash equilibrium, let's construct a payoff matrix with the ex-ante expected payoffs:

	M	P
RR	0, 0	1, -1
RF	0.5, -0.5	0, 0
FR	-0.5, 0.5	1, -1
FF	0, 0	0, 0

Notice that FF is strictly dominated and FR is weakly dominated, so it seems likely that the equilibrium we are looking involves mixing just between RR and RF . Let's denote the probability that player 1 plays RR by σ_1 , then player 2 is indifferent between M and P , if

$$\sigma_1 \cdot 0 + (1 - \sigma_1) \cdot (-0.5) = \sigma_1 \cdot (-1) + (1 - \sigma_1) \cdot 0$$

Solving for σ_1 yields $1/3$. Let's now find the strategy of player 2. Denote by σ_2 the probability that player 2 plays M , then player 1 is indifferent between RR and RF if

$$\sigma_2 \cdot 0 + (1 - \sigma_2) \cdot 1 = \sigma_2 \cdot 0.5 + (1 - \sigma_2) \cdot 0$$

which gives $\sigma_2 = 2/3$. Thus according to the strategies we derived, player 1 plays RR with probability $1/3$ and RF with probability $2/3$ and player 2 plays M with probability $2/3$ and P with probability $1/3$. It's easy to see that this is indeed an equilibrium, since player 1 would get strictly less from playing either FR or FF against player 2's strategy.

- (c) Write the corresponding behavior strategies (i.e. the behavior strategies generated by the equilibrium mixed strategy profile).

Solution. We need to recall here that a mixed strategy is a probability distribution over strategies whereas a behavior strategy is a probability distribution over actions at each history. Since the game here is relatively simple, this is fairly straightforward. Player 2 is only playing at one information set and is randomizing over her actions M and P . We already found the probabilities, i.e. $b_2(M) = 2/3$ and $b_2(P) = 1/3$. Player 1 is always raising with red (playing either RR or RF) and raising on black with probability $1/3$, so we can write the behavior strategy as $b_1(R|red) = 1$, $b_1(F|red) = 0$, $b_1(R|black) = 1/3$ and $b_1(F|black) = 2/3$.

- (d) Derive a belief system (probabilities for withing each information set) that is consistent with the equilibrium strategies (i.e. derived using the Bayesian rule).

Solution. We need to derive the beliefs for player 2 about the color of the card, when she is making a decision whether to meet or pass given player 1 strategy. Let's denote the conditional probability of the card being red after a raise by $\mu(red|R)$ and black by $\mu(black|R)$. By Bayes' rule:

$$\begin{aligned}\mu(red|R) &= \frac{0.5 \cdot 1}{0.5 \cdot 1 + 0.5 \cdot (1/3)} = 0.75 \\ \mu(black|R) &= \frac{0.5 \cdot (1/3)}{0.5 \cdot 1 + 0.5 \cdot (1/3)} = 0.25\end{aligned}$$

- (e) Check that the equilibrium strategies are sequentially rational given the belief system that you derived in (d).

Checking that the strategies are sequentially rational means that we check whether the players would be willing to play according to the equilibrium strategies at each of their information sets given the other players' strategy.

Player 1:

Red card: R yields $(2/3) \cdot 2 + (1/3) \cdot 1 > 1 =$ payoff from F. Black card: R yields $(2/3) \cdot -2 + (1/3) \cdot 1 = -1 =$ payoff from F.

Since player 1 cannot do any better against player 2's strategy in either of the two information sets, her strategy is sequentially rational.

Player 2:

We need to use the belief probabilities here. After a raise, the card is red with probability 0.75 and black with 0.25. Thus M yields: $0.75 \cdot (-2) + 0.25 \cdot 2 = -1$, which equals the payoff from P, which is always -1 . Player 2 is indifferent and mixing is rational.

5. An entrant firm (player 1) decides whether to enter an industry with an incumbent firm (player 2). Entry costs $c = 1$. If there is no entry, then player one gets payoff of 0 and player 2 gets a payoff of 3. If there is entry, then the firms decide simultaneously whether to fight or cooperate with payoffs given in the matrix below (so that the total payoff of player one is the payoff given in the matrix minus her entry cost):

	F	C
F	-1, 0	0, -1
C	0, 0	2, 2

- (a) Define the extensive form game.

Solution. The game can be formulated with two different extensive forms: one where player 1's information set will be first after entry and another where player 2's information set will be first. Both will be drawn in class. We will also cover the case where player 1 chooses simultaneously her entry (E/N) and the type of the entry (F/C), leading to three actions in the first node: N, EF, and EC.

- (b) Find all Nash equilibria.

Solution. Let's look at the strategic form of the game:

	F	C
NF	0, 3	0, 3
NC	0, 3	0, 3
EF	-2, 0	-1, -1
EC	-1, 0	1, 2

There are three pure strategy Nash equilibria: (NF,F), (NC,F) and (EC,C).

- (c) Find all subgame perfect Nash equilibria.

Solution. The equilibrium in last stage game is (C,C). Thus the equilibrium of the whole game is (EC,C).

- (d) Find all weak perfect Bayesian equilibria.

Solution.

In a Bayesian equilibrium the payoffs of the the agents are evaluated using their beliefs about the previous play of the game. More precisely, the weak perfect Bayesian equilibrium (PBE) requires that the beliefs are derived using the Bayes' rule wherever applicable, i.e. on the equilibrium path. Off-equilibrium path beliefs can be arbitrary, since the probability of reaching the information set off-equilibrium is zero. Because of this the equilibrium in the game we specified above depends on the order in which we put the players in the extensive form.

Let's have look at the payoffs of the players as a function of their beliefs. Let μ_i denote player's i belief that she is in the node in which player j has played F . Furthermore, let $U_i(a, \mu_i)$ denote the payoff for player i from playing a . Then the payoffs from different actions for the players are:

$$U_1(F, \mu_1) = -\mu_1 - 1; U_1(C, \mu_1) = 2(1 - \mu_1) - 1 \Rightarrow U(C) > U(F) \forall \mu_1 \in [0, 1].$$

$$U_2(F, \mu_2) = 0; U_2(C, \mu_2) = -\mu_2 + 2(1 - \mu_2) \Rightarrow U(C) \geq U(F) \text{ iff } \mu_2 \leq \frac{2}{3}.$$

So player 1 is always at least as well off from playing C than F whatever her belief. Player 2 will play F if her belief that player 1 plays F is greater than $2/3$. This means that in the game where player 1's information set comes first, we can support an equilibrium where player 1 does not enter by specifying player 2's belief as $\mu_2 > (2/3)$. Since player 2's information set is reached with probability 0 in this equilibrium, we are not restricted in the way we can specify these beliefs. Similar argument does not hold when player 2 has her information set first, since playing F is not sequentially rational for any belief because it is strictly dominated by C . (Player 2 has no beliefs (single node) and evaluates the payoffs given the equilibrium strategies.)

If player 1 moves first, there is a PBE with behavior strategies with the beliefs μ_2 as above

$$b = ((b_1(N) = 1, b_1(E) = 0, b_1(F) = 0, b_1(C) = 1), (b_2(F) = 1, b_2(C) = 0))$$

and another one with

$$b = ((b_1(N) = 0, b_1(E) = 1, b_1(F) = 0, b_1(C) = 1), (b_2(F) = 0, b_2(C) = 1))$$

. If player 2 moves first, only the latter strategy profile constitutes a PBE. Are the two situations strategically different?

(e) Find all sequential equilibria.

Solution. There is only one sequentially rational equilibrium, $b = ((b_1(N) = 0, (b_1(E) = 1), (b_1(F) = 0), (b_1(C) = 1)), ((b_2(F) = 0), (b_2(C) = 1)))$, even in the extensive form where player 1's decision node comes first. This is because in a sequentially rational equilibrium the beliefs, μ , must be derived from some sequence of strategies that converges to the equilibrium strategies. Beliefs derived this way are called *consistent*.

Let's show that staying out cannot happen in a SE and specify two arbitrary sequences ϵ_0^n and ϵ_1^n , which converge 0, and use these to write a sequence of the player 1's behavior strategies $b^n = (b_1(N) = 1 - \epsilon_0^n, b_1(E) = \epsilon_0^n, b_1(F) = \epsilon_1^n, b_1(C) = 1 - \epsilon_1^n)$. Player 2's beliefs derived from these sequences are:

$$\mu_2(F) = \frac{\epsilon_0^n \epsilon_1^n}{\epsilon_0^n \epsilon_1^n + \epsilon_0^n (1 - \epsilon_1^n)} = \epsilon_1^n \rightarrow 0$$

$$\mu_2(C) = 1 - \mu_2(F) \rightarrow 1$$

Thus the beliefs we specified in finding the PBE, in which player 1 does not enter are not consistent. Hence, the only sequentially rational equilibrium of the game is the one where 1 enters and both players cooperate.

Consider the game, where player 1 chooses simultaneously her entry and its type. Show that (N, F) is a SE! Is this game strategically different from the original one?

6. Two players are contributing to a public good over time. Player 1 contributes in odd periods and player 2 in even periods. If player i contributes in period t amount z_{it} she bears an individual cost $c_i(z_{it}) = z_{it}$. All past contributions are irreversible and publicly observable. Once the total cumulative contribution exceeds a threshold $\bar{z}$, both players get a one time payoff π and the game is over. The players maximize their payoff net of their individual cost of providing the public good. Assume that $\pi < \bar{z} < 2\pi$.

- (a) For the case where $t \in \{1, 2\}$, find the subgame perfect equilibria of the game. Are there other Nash equilibria?

Solution. One way to look at this game is to see it as a bargaining game, i.e. the players are trying to divide the surplus, $2\pi - \bar{z}$, from the public good between themselves. Since there is no discounting, any such strategy profile that gives both players at least their cumulative contribution, $Z_i = \sum_{t=0}^{\infty} z_i$, is a candidate for an equilibrium.

In (a) there are only two periods, which gives player 1 an advantage. Let's solve for the subgame perfect equilibrium using backward induction: what is the maximum amount that player 2 is willing to contribute in period 2? Completing the project gives her π so contributing $\hat{Z}_2 = \pi$ makes her indifferent between contributing and not contributing and thus is the maximum amount she is willing to contribute. Thus in the first round player 1 should contribute $\hat{Z}_1 = \bar{z} - \pi$. This is the subgame perfect equilibrium of the game.

Are there other Nash equilibria? Any such strategies, where player 1 contributes $\hat{Z}_1 \leq \pi$ and player 2 contributes $\hat{Z}_2 \leq \pi$ and for which $\hat{Z}_1 + \hat{Z}_2 = \bar{z}$ are Nash equilibria, because if we are looking the game from the perspective of period $t = 1$ there are no profitable deviations for either of the players. Strategies, leading to this outcome, take the following form:

$$z_1 = \hat{Z}_1.$$

$$z_2 = \begin{cases} \hat{Z}_2 & \text{if } z_1 = \hat{Z}_1 \\ 0 & \text{otherwise.} \end{cases}$$

Also $(0, 0)$ is a NE but not a SPE.

- (b) The same questions with $t \in \{1, 2, \dots, T\}$.

Solution. We can again use backward induction to find the subgame perfect equilibrium: what is the maximum amount that player i is willing to contribute in period T ? Clearly the answer has to be the same as in the previous case, $\hat{Z}_i = \pi$. If T is odd this will be player 1 and if it is even it will be player 2. Thus all strategy profiles in which the last player to play contributes $\hat{Z}_i = \pi$ and the other player $\hat{Z}_j = \bar{z} - \pi$ are subgame perfect equilibria. The timing of the contributions does not matter, because there is no discounting. However, the player moving in $T - 1$ has to contribute the whole amount $\hat{Z}_j$ before the other player starts to contribute.

The same strategies described in the previous case are Nash equilibria here as well, but of course the timing of the contributions can now vary.

- (c) Assume that the time horizon is infinite. What kind of sub-game perfect equilibria can you find?

Solution. The previous backward induction argument does not work as the players can always opt to wait. Thus now some of previous Nash equilibrium strategy profiles are subgame perfect: player 1 contributes in total $\hat{Z}_1 \leq \pi$ and player 2 contributes in total $\hat{Z}_2 \leq \pi$ with $\hat{Z}_1 + \hat{Z}_2 = \bar{z}$. To see how a strategy profile like this is a subgame perfect equilibrium fix player 2's strategy as to contribute $\hat{Z}_2$ iff at least the amount of $\hat{Z}_1$ has been contributed before. The candidate for player 1's equilibrium strategy is to contribute $\hat{Z}_1 - Z$, where Z is the total amount of contributions in the previous periods. Does player 1 have a profitable deviation after $\hat{Z}_1$ has been contributed? No, since she knows that player 2 will contribute. Does player 1 have a profitable deviation before $\hat{Z}_1$ has been contributed? Clearly not, since given player 2 strategy the only way the public good will be produced is that player 1 contributes up to $\hat{Z}_1$. She does not want to contribute more than that because player 2 will do the rest. Similarly, player 2 has no profitable deviations. All such strategy profiles that are based on this sort of cutoffs for other player's contribution and for which the investment level will equal $\bar{z}$ are SPE.

Note that we cannot use OSDP here since payoffs are not discounted and hence distant future never becomes irrelevant. Fortunately, equilibrium strategies are almost stationary and it is relatively easy to check every kind of deviations.