FDPE Microeconomics 3: Game Theory
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Lecture notes 2: Sequential rationality in dynamic games
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1 Outline

- We now move to dynamic games
- We focus especially on Nash equilibrium refinements induced by sequential rationality: sub-game perfect equilibrium and sequential equilibrium
- Material: MWG Chapter 9
- Other relevant sources: Fudenberg-Tirole Ch. 3-4, 8.3, Osborne-Rubinstein Ch. 6-7, 12, Myerson Ch. 4
- Some motivating examples:

1.1 Example: predation

- An entrant considers entry into an industry with a current incumbent firm
- Entry costs 1 unit
- Monopoly profit in the industry is 4
- If entry takes place, the monopolist can either accomodate or fight
- Accommodation splits monopoly profits, whereas fighting gives zero profit to both firms
- Will entrant enter, and if so, will incumbent fight or accomodate?
- Normal form representation of the game:

	Fight if entry	Accommodate if entry
Enter	$-1, 0$	$1, 2$
Stay out	$0, 4$	$0, 4$

- There are two Nash equilibria: (Enter, Accommodate) and (Stay out, Fight if entry)
- Is one of the two equilibria more plausible?

1.2 Example: quality game

- A producer can produce an indivisible good, and choose either high or low quality
- Producing high quality costs 1 and bad quality 0
- Buyers values high quality at 3 and bad quality at 1
- For simplicity, suppose that good must be sold at fixed price 2
- Which quality will be produced and will the buyer buy?
- Normal form representation of the game:

	High quality	Low quality
Buy	$1, 1$	$-1, 2$
Do not buy	$0, -1$	$0, 0$

- Only one Nash equilibrium (Do not buy, Low)
- What if seller moves first?
- What if buyer moves first?
- What if seller moves first, but quality is unobservable?

1.3 Example: Stackelberg vs. Cournot

- Consider the quantity setting duopoly with profit functions

$$\pi_i\left(q_i, q_j\right) = q_i(1 - q_1 - q_2), \text{ for all } i = 1, 2.$$

- Suppose the players set their quantities simultaneously (Cournot model). The unique Nash equilibrium is:

$$(q_1^*, q_2^*) = \left(\frac{1}{3}, \frac{1}{3}\right),$$

  which gives payoffs

$$\pi_1\left(\frac{1}{3}, \frac{1}{3}\right) = \pi_2\left(\frac{1}{3}, \frac{1}{3}\right) = \left(1 - \frac{2}{3}\right)\frac{1}{3} = \frac{1}{9}.$$

- What if player 1 moves first? (Stackelberg model)
- After observing the quantity choice of player 1, player 2 chooses his quantity.

- Given the observed q_1, firm 2 chooses q_2. Optimal choice is

$$BR_2(q_1) = \frac{1-q_1}{2}.$$

- Player 1 should then choose:

$$\max_{q_1} u_1(q_1, BR_2(q_1)) = \left(1 - q_1 - \frac{1-q_1}{2}\right) q_1 = \frac{(1-q_1)q_1}{2}.$$

- This leads to

$$q_1 = \frac{1}{2},\ q_2 = BR_2(q_1) = \frac{1}{4}$$

  with payoffs

$$\pi_1\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{8},\ \pi_2\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{16}.$$

1.4 Example: matching pennies

	Head	*Tail*
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- Nash equilibrium, where both players mix with 1/2 probabilities
- What if player 1 moves first?
- What if player 1 moves first, but choice is unobservable?

1.5 Discussion

- These examples illustrate that the order of moves is crucial
- Moving first may help (commitment to an action)
- Moving first may also hurt (matching pennies)
- The normal form representation misses the dynamic nature of events, so we need to utilize extensive form representation
- The key principle is *sequential rationality*, which means that a player should always use a continuation strategy that is optimal given the current situation
- For example, once entrant has entered, the incumbent should act optimally given this fact (accommodate)
- This will lead us to refinements of Nash equilibrium, in particular *subgame perfect equilibrium (SPE)* and *sequential equilibrium*

2 Subgame perfect equilibrium

2.1 Subgame

- Consider an extensive form game with perfect recall
- A subgame is a subset of the original game-tree that inherits information sets and payoffs from the original game, and which meets the following requirements:

    1. There is a single node such that all the other nodes are successors of this node (that is, there is a single initial node to the subgame)
    2. Whenever a node belongs to the subgame, then all of its successor nodes must also belong to the subgame.
    3. Whenever a node belongs to the subgame, then all nodes in the same information set must also belong to the subgame.

2.2 Subgame perfect equilibrium

Definition 1 *A strategy profile σ of an extensive form game is a subgame perfect Nash equilibrium (SPE) if it induces a Nash equilibrium in every subgame of the original game.*

- Every subgame perfect equilibrium is a Nash equilibrium, but the converse is not true. Thus, subgame perfection is a *refinement* of the Nash equilibrium concept.
- The idea is to get rid of equilibria that violate sequential rationality principle.
- It is instructive to go through the examples that we've done so far and identify subgame perfect equilibria.

2.3 Backward induction in games of perfect information

- In finite games with perfect information, sub-game perfect equilibria are found by backward induction:
- Consider the nodes, whose immediate successors are terminal nodes
- Specify that the player who can move in those nodes chooses the action that leads to the best terminal payoff for her (in case of tie, make an arbitrary selection)
- Then move one step back to the preceding nodes, and specify that the players who move in those nodes choose the action that leads to the best terminal payoff - taking into account the actions specified for the next nodes

- Continue this process until all actions in the game tree have been determined
- This process is a multi-player generalization of *backward induction* principle of dynamic programming

Theorem 2 *A finite game of perfect information has a subgame perfect Nash equilibrium in pure strategies.*

- The proof is the backward induction argument outlined above
- If optimal actions are unique in every node, then there is a unique subgame perfect equilibrium
- Note that the terminal nodes are needed to start backward induction. Does not work for infinite games.
- How about chess?

2.4 Example: chain store paradox

- Note that the entry game (predation) discussed at the beginning of the lecture is a perfect information game
- Could a firm build a reputation for fighting if it faces a sequence of entrants?
- *Chain store paradox* considers an incumbent firm CS that has branches in cities $1, ..., K$.
- In each city there is a potential entrant.
- In period k, entrant of city k enters or not. If it enters, incumbent may fight or accomodate.
- Payoffs for city k are as in original entry game:

	Fight if entry	Accommodate if entry
Enter	$-1, 0$	$1, 2$
Stay out	$0, 4$	$0, 4$

- Incumbent maximizes the sum of payoffs over all cities, while each entrant maximizes profits of that period.
- An entrant only enters if it knows the CS does not fight.
- Would it pay for CS to build a reputation of toughness if K is very large?
- The paradox is that in SPE, the CS can not build a reputation.
- In the final stage, the optimal action of CS is Accomodate, if the entrant enters.

- The entrant know this, and thus enters.
- By backward induction, this happens in all stages.
- We find that the unique SPE is that all entrants enter and CS always accomodates.
- To bring reputation effects to life, we would need to introduce *incomplete* information (later in this course)

2.5 Example: centipede game

- Centipede game is a striking example of backward induction
- Two players take turns to choose Continute (C) or stop (S)
- The game can continue at most K steps (K can be arbitrarily large)
- In stage 1, player 1 decides between C and S. If he chooses S, he gets 2 and player 2 gets 0. Otherwise game goes to stage 2.
- In stage 2, player 2 decides between C and S. If he chooses S, he gets 3 and player 2 gets 1. Otherwise game goes to stage 3, and so on.
- If i stops in stage k, he gets $k+1$, while j gets $k-1$.
- If no player ever stops, both players get K.
- Draw extensive form and solve by backward induction. What is the unique SPE?

3 Multi-stage games with observed actions

- One restrictive feature of games of perfect information is that only one player moves at a time
- A somewhat larger class of dynamic games is that of multi-stage games with observed actions
- Many players may act simultaneously within each stage
- We may summarize each node that begins stage t by history h^t that contains all actions taken in previous stages: $h^t := \left(a^0, a^1, ..., a^{t-1}\right)$
- A pure strategy is a sequence of maps s_i^t from histories to actions $a_i^t \in A_i\left(h^t\right)$
- Payoff u_i is a function of the terminal history h^{T+1}

3.1 One-step deviation principle

- Since many players may act simultaneously within a stage, backward induction argument can not be applied as easily as with games of perfect information
- However, the following principle that extends backward induction idea is useful:

Theorem 3 *In a finite multi-stage game with observed actions, strategy profile s is a subgame perfect equilibrium if and only if there is no player i and no strategy s_i' that agrees with s_i except at a single t and h^t, and such that s_i' is a better response to s_{-i} than s_i conditional on history h^t being reached.*

- That is: to check if s is a SPE, we only need to check if any player can improve payoffs by a one-step deviation
- Note that the result requires a finite horizon, just like backward induction
- Some applications have an infinite horizon in which case payoffs defined as functions of the infinite sequence of actions
- Importantly, the result carries over to such games under an extra condition that essentially requires that distant events are relatively unimportant
- In particular, if payoffs are discounted sums of per period payoffs, and payoffs per period are uniformly bounded, then this condition holds
- The proof of the one-step deviation principle is essentially the principle of optimality for dynamic programming.

3.2 Example: repeated prisoner's dilemma

- Consider prisoner's dilemma with payoffs:

	Cooperate	*Defect*
Cooperate	$1, 1$	$-1, 2$
Defect	$2, -1$	$0, 0$

- Suppose that two players play the game repeatedly for T periods, with total payoff

$$\frac{1-\delta}{1-\delta^T}\sum_{t=0}^{T-1}\delta^t g_i\left(a^t\right),$$

  where g_i gives the per-period payoff of action profile a^t as given in the table above
- Players hence maximize their discounted sum of payoffs, where the term $\frac{1-\delta}{1-\delta^T}$ is just a normalization factor to make payoffs of games with different horizons easily comparable

- Suppose first that the game is played just once ($T = 1$). Then (Defect,Defect) is the unique Nash equilibrium (Defect is a dominant action)
- Suppose next that T is finite. Now, subgame perfection requires both players to defect in the last period, and backward induction implies that both players always defect.
- Finally, suppose that T is infinite. Then backward induction cannot be applied but one-step deviation principle holds (discounting and bounded payoffs per period)
- "Both defect every period" is still a SPE
- However, provided that δ is high enough, there are now other SPEs too
- By utilizing one-step deviation principle, show that the following is a SPE: "cooperate in the first period and continue cooperating as long as no player has ever defected. Once one of the players defect, defect in every period for the rest of the game".

4 Sequential equilibrium

- Recall that a subgame starts with an information set that consists of a *single* node
- But in games of imperfect information, there may be few such nodes
- For example, in Bayesian games, where nature chooses a type for each player, the only subgame is the whole game tree
- In such situations the refinement of subgame perfect equilibrium has no bite
- To evaluate sequential rationality in information sets with many nodes, we must consider the *beliefs* of the player that chooses her action
- We define a *belief system* as:

Definition 4 *A belief system μ assigns for each information set h a probability distribution on the nodes of that information set. In other words, $\mu^h(x) \in [0,1]$ gives a probability of node x in information set h, where $\Sigma_{x \in h} \mu^h(x) = 1$.*

- In words, μ^h expresses the beliefs of player $\iota(h)$ on the nodes in h conditional on reaching h.
- Let b denote some behavior strategy

- Let $\mathbf{a}^x$ be the path of actions that leads from x^0 to x. Define

$$P^b(x) = \prod\{b(a)|a \in \mathbf{a}^x\},$$

and

$$P^b(h) = \sum\{P^b(x)|x \in h\}.$$

- Let $u_i(b|\mu^h)$ be the expected utility of player i given that information set h is reached, given that player i's beliefs with respect to the nodes $x \in h$ is given by μ^h, and given that the strategy profile b is played on all information sets that follow h

- Sequential rationality can now be formally stated:

Definition 5 *behavior strategy profile b is sequentially rational (given belief system μ) if for all i and all h such that i moves at h,*

$$u_i(b|\mu^h) \geq u_i((a_i, b_i^{-h}), b_{-i}|\mu^h) \text{ for all } a_i \in A_i(h).$$

- By one-step deviation principIe, this definition implies expected utility maximization at each h given the beliefs at h and given that all future decisions are taken according to b.

- So far we have said nothing about how beliefs are formed

- To connect beliefs to strategies, we require that they are obtained from the strategies using Bayes' rule:

$$\mu^h(x) = \frac{P^b(x)}{P^b(h)}, \text{ whenever } P^b(h) > 0.$$

- We have:

Definition 6 *A Perfect Bayesian Equilibrium (PBE) is a pair (b, μ) such that b is sequentially rational given μ and μ is derived from b using Bayes' rule whenever applicable.*

- What to do with off-equilibrium pats, i.e. information sets such that $P^b(h) = 0$?

- The version of Perfect Bayesian Equilibrium defined above gives full freedom for choosing those beliefs (this version is called *weak* PBE in MWG)

- Why do off-equilibrium beliefs matter? Because they may induce off-equilibrium actions that in turn influence behavior *on* the equilibrium path

- To make the concept of PBE more useful in applications, additional restrictions for off-equilibrium beliefs have been introduced (see e.g. Fudenberg-Tirole section 8.2, or MWG section 13.C), but this is not a general cure as it may lead to non-existence problems

- The solution concept, introduced in Kreps and Wilson (1982, Econometrica), called *sequential equilibrium* derives beliefs at off-equilibrium information sets as limits from strategies that put a positive but small probability on all actions (so that all information sets are reached with positive probability):

  Definition 7 *A pair (b, μ) is a Sequential Equilibrium if:*

  1) Sequential Rationality: b is sequentially rational given μ

  2) Consistency of beliefs: there exists a sequence of pairs $(b^n, \mu^n) \to (b, \mu)$, such that for all n, b^n puts a positive probability on all availabe actions, and for any h and any $x \in h$, $\mu_h^n(x) = P^{b^n}(x)/P^{b^n}(h)$.

- Every finite extensive form game with perfect recall has a sequential equilibrium

- In practice, PBE is a popular solution concept in applications

- Sequential equilibrium is important because:

  - Existence is guaranteed (in finite games with perfect recall)
  - Every sequential equilibrium is at the same time a (weak) perfect Bayesian equilibrium
  - Also, if (b, μ) is a sequential equibrium, then at the same time b is a sub-game perfect equilibrium (this does not necessarily hold for a weak PBE).

- A related concept is called extensive form trembling-hand perfect Nash equilibrium, which also always exists in finite games (see MWG Appendix B to Ch. 9). An extensive form trembling-hand perfect equilibrium is a sequential equilibrium, but the converse is not necessarily true.