PAULI MURTO, ANDREY ZHUKOV

Introduction

If any mistakes or typos are spotted, kindly communicate them to andrey.zhukov@aalto.fi.

Materials from Osborne and Rubinstein (1994), Battigalli (2017) and Maschler, Solan and Zamir (2013) were used in preparation of the problem set. I (Andrey) particularly encourage you to consult Battigalli (2017).

Problems

In problem 1, we investigate what predictions about the play of the game we can make based solely on the knowledge that one of the players is rational. We also study how the quality of such predictions varies with risk attitude of the player.

Problem 1 (Risk-aversion and implications of rationality)

Consider the following static game:

1. *Set of players,* $\mathcal{I} = \{1, 2\}$
2. *Pure strategy space for player* 1*,* $S_1 = \{t, m, b\}$
3. *Pure strategy space for player* 2*,* $S_2 = \{l, r\}$
4. *An outcome function maps a strategy profile to monetary payments for player* 1 *and player* 2*:* $g : S_1 \times S_2 \to \mathbb{R}^2$*. Player* 1*'s component of the outcome function,* $g^1 : S_1 \times S_2 \to \mathbb{R}$ *is presented in table 1 below*

Further, we assume that player 1 *is rational. Our purpose here is to investigate implication of player* 1*'s rationality on her play, and how the implications vary*

	l	r
t	3	0
m	1	1
b	0	3

Table 1: Monetary payment for player 1 as a function of the strategy profile

with her risk attitude. [1]

a) Assume that player 1 is risk-neutral and self-centered, that is her von Neumann-Morgenstern utility function, $v : \mathbb{R}^2 \to \mathbb{R}$, is defined by $v(x, y) = x$, where the first argument of the function v is monetary payment to player 1, and the second argument is monetary payment to player 2. Which pure strategies of player 1 are never best response? In other words, with the assumptions we made on player 1's preferences over monetary outcomes and lotteries over them, the play of which strategies contradicts her rationality?

b) Assume that player 1 is risk-averse and self-centered, with her von Neumann-Morgenstern utility function, $v : \mathbb{R}^2 \to \mathbb{R}$, defined by $v(x, y) = \sqrt{x}$, where the first argument of the function v is monetary payment to player 1, and the second argument is monetary payment to player 2. Which pure strategies of player 1 are never best response? In other words, with the assumptions we made on player 1's preferences over monetary outcomes and lotteries over them, the play of which strategies contradicts her rationality?

c) Compare your answers to parts *a)* and *b)*. As player 1 becomes risk-averse, does the set of justifiable strategies of player 1 expands, shrinks or does not change? [2] Does the set of the outcomes of the play consistent with rationality of player 1 expands, shrinks, or does not change as she becomes more risk-averse? What is the intuition behind that? Do you think it is a general result or it is specific to the game we analyzed, and why?

[1]**Reminder**: if we know that a player is rational, all we could say about her play is that she is not going to play a strategy which is never best response. Recall, that a strategy s_1 of player 1 is never best response if there is no conjecture, $\mu \in \Delta(S_2)$, player 1 could make on the actions of player 2, such that s_1 is a best-response to μ.

[2]A strategy of player 1, s_1, is justifiable if it is not never best response

Just knowing that players are rational often does not let us make any predictions on the play apart from “anything can happen". Therefore game theorists typically assume that players are not only rational, but there is also common knowledge of rationality between the players. In problem 2, we will investigate implications of assuming common knowledge of rationality rather than “only" rationality.[3]

Problem 2 (Common knowledge of rationality and its implications)

Consider the game $<\{1,2\},(S_i),(u_i)>$*, where* $S_i := \mathbb{R}^+$ *and* $u_i(s_1, s_2) = s_i * \max\{0, \frac{\alpha}{\beta} - \frac{s_1}{2\beta} - \frac{s_2}{2\beta}\} - \frac{s_i^2}{2m}$*, where* $\alpha > 0$*,* $\beta > 0$*, and* $m > 0$*. Notice that this captures Cournot duopoly game, with the demand function* $D(p) = 2\max\{0, \alpha - \beta p\}$ *and the cost function* $C(s_i) = \frac{s_i^2}{2m}$*. What are implications of assuming common knowledge of rationality?*[4]

In any game, a rational player is not going to play a strategy which is never best response. Trying to verify that there is no conjecture which would justify the strategy could be cumbersome, and therefore a natural question is whether we could characterize the set of justifiable actions with no reference to conjectures and expected payoff maximization. In the lecture notes, this question was answered positively for a general game - it was proven that a strategy, s_i, is never best reply (not justifiable) if and only if it is strictly dominated. Problem 3 provides intuition for the proof presented in the lecture notes with the particular example.

Problem 3 (Intuition for the separating hyperplane proof)

Consider the following static game:

[3] Also see extra-problem 1 where we informally attempt to clarify the concept of common knowledge and contrast it with the concept of mutual knowledge

[4] Hint: assuming common knowledge of rationality allows us to iteratively delete strictly dominated strategies

1. *Set of players,* $\mathcal{I} = \{1, 2\}$

2. *Pure strategy space for player* 1*,* $S_1 = \{a, b, c, d\}$

3. *Pure strategy space for player* 2*,* $S_2 = \{l, r\}$

4. *von Neumann-Morgenstern utility function of player* i*,* $u_i : S_1 \times S_2 \to \mathbb{R}$*.* *Player* 1*'s von Neumann-Morgenstern utility function,* $u^1 : S_1 \times S_2 \to \mathbb{R}$ *is presented in the matrix below*

	l	r
a	5	1
b	1	5
c	2	2
d	1	1

Any conjecture player 1 *might have on the play of* 2 *induces preference relation on the space of payoff pairs. Plot the graph where (i) horizontal axis represents payoff to player* 1 *when* 2 *plays* l*; (ii) vertical axis represents payoff to player* 1 *when* 2 *plays* r*.*

a) Plot the extreme points of the set of feasible expected payoff vectors of player 1 which is defined below

$$\mathbf{E} := \{(x, y) : \exists s_1 \in S_1 : x = u(s_1, l), y = u(s_1, r)\}$$

(Definition in words: the set of extreme points is the set of (x, y)-pairs such that player 1 has a strategy such that when this strategy is played and 2 plays l, 1's payoff is x, and such that when this strategy is played and 2 plays r, 1's payoff is y

b) Plot the set of feasible payoff vectors for player 1 (in other words, the set of all convex combinations of the vectors plotted in part *a)*)

c) In the set of feasible payoff vectors, mark the payoff vectors which correspond to the strategies which are not dominated

d) Pick any payoff vector corresponding to the strategy which is not dominated, and plot the set of vectors in $\mathbb{R}^2$ which are strictly greater than the chosen payoff vector [5]

e) Plot a separating hyperplane - a line separating the set of feasible payoffs from the set plotted in part *d)* of the problem

f) Plot the vector normal to the separating hyperplane

g) Divide each coordinate of the normal vector by its norm

h) Show that the chosen strategy is the best-response to the conjecture corresponding to the normalized normal vector

As we saw in the Cournot competition example of the lecture notes or in problem 2 above, sometimes all but one conjecture on the play of others contradict assumption of common knowledge of rationality between the players in the game - and in such cases, we can make precise prediction on the outcome of the play. However this is not the case in most of the interesting games about which we still want to be able to say something of essence. In those games, a natural way to fix conjectures of the players about the play of others is correlated equilibrium demonstrated in problem 4 below.

Problem 4 (Correlated equilibrium)

Consider the following two-player game:

	l	r
t	$5,1$	$0,0$
b	$4,4$	$1,5$

a) What is the set of rationalizable strategies in the game above?

b) Find all Nash equilibria of the game. What is the best payoff players can get in the symmetric equilibrium?[6]

c) Suppose that before choosing their actions, the players first toss a coin. After publicly observing the outcome of the coin toss, they

[5]Strictly greater: =greater in each coordinate
[6]In the symmetric equilibrium expected payoffs of the players must be the same

choose simultaneously their action. Draw the extensive form of the described game and define available strategies for the players. Find a symmetric Nash equilibrium that gives both players a higher payoff than the symmetric equilibrium in *a)*.

d) Suppose that there is a mediator that can make a recommendation separately and covertly for each player. Suppose that the mediator makes recommendation $(\mathtt{t}, \mathtt{l})$, or $(\mathtt{b}, \mathtt{l})$, or $(\mathtt{b}, \mathtt{r})$, each with probability $\frac{1}{3}$. Each player only observes her own action choice recommendation (so that, e.g., the row player upon seeing the recommendation $\mathtt{b}$ does not know whether the recommended profile is $(\mathtt{b}, \mathtt{l})$ or $(\mathtt{b}, \mathtt{r})$. Does any of the players have an incentive to deviate from the recommended action? What is the expected payoff under this scheme?

Problem 5 (Zero-sum game)

Consider the following two-player zero-sum game.

	l	c	r
t	3	-3	0
m	2	6	4
b	2	5	6

Table 2: Payments player 2 makes to player 1

a) Find a mixed strategy of player 1 that guarantees him the same payoff against any pure strategy of player 2.

b) Find a mixed strategy of player 2 that guarantees him the same payoff against any pure strategy of player 1.

c) Does the play of the mixed strategies found above constitutes Nash equilibrium of the game?

Extra problems

Extra problem 1 (Mutual vs. Common Knowledge)

There are ten citizens living in the land of Logos behind the Iron Curtain. Three of them were born with a stamp saying "irrational" on their foreheads, while the rest has no stamps. The law of Logos is such that it prohibits citizens to talk about the forehead stamps. Therefore every citizen knows whether another citizen is stamped or not, but does not know it about himself. Since any discussions on the topic are prohibited, the statistics on the number of "irrational" citizens is also unavailable. Further, if a citizen finds out that he is stamped "irrational", his conscience will force him to leave the land of Logos the next day after he learns it. Citizens obey the law, and are highly deductive in the sense that any conclusion which can be reached by a logically consistent argument of any difficulty is automatically known by the citizens. One day, a rock star from beyond the Iron Curtain visits the land of Logos and in its address to the whole population of the land remarks that it is surprising to see a person stamped "irrational" in this land. Prove (for instance, by induction on the number of citizens stamped "irrational") that 3 *days after an address of the rock-star all "irrational" citizens will leave the land of Logos. [**Hint:** Suppose there is only one stamped citizen in the land. Then, what does he do a day after the rock-star's address?] Everyone could see that there was at least one "irrational" citizen even before the rock-star's address, that is it was mutual knowledge that there is at least one "irrational" citizen. Then what did the announcement change?*

Extra problem 2

Consider the following three-player game with action space $A = A_1 \times A_2 \times A_3 = \{U, D\} \times \{L, R\} \times \{M_1, M_2, M_3, M_3\}$. *In the representation below, the first player chooses the row, the second player chooses a column and the third player chooses a matrix. Utility of each action profile is the same across players.*

M1

	L	*R*
U	*8*	*0*
D	*0*	*0*

M2

	L	*R*
U	*0*	*0*
D	*0*	*8*

M3

	L	*R*
U	*4*	*0*
D	*0*	*4*

M4

	L	*R*
U	*3*	*3*
D	*3*	*3*

a) Show that M_3 is not strictly dominated *b)* Can you find independent conjectures on the play of 1 and 2 such that M_3 is best response to such conjectures? [7]

c) Now allow for correlated conjectures, and find a correlated conjecture to which playing M_3 is best response.

Extra problem 3 (Partnership game)

Lee (Player 1), and Julie (Player 2), are business partners. Each of the partners has to determine the amount of effort he or she will put into the business, which is denoted by e_i, $i \in \{1, 2\}$, *and may be any nonnegative real number. The cost of effort* e_i *for Player* i *is* ce_i, *where* $c > 0$ *is equal for both players. The success of the business depends on the amount of effort put in by the players; the business's profit is denoted by* $r(e_1, e_2) = e_1^{\alpha_1} e_2^{\alpha_2}$, *where* α_1, $\alpha_2 \in (0, 1)$ *are fixed constants known by Lee and Julie, and the profit is shared equally between the two partners. Each player's utility is given by the difference between the share of the profit received by that player and the cost of the effort he or she put into the business.*

a) Describe this situation as a game in the strategic form. Note that the set of strategies of each player is the continuum.

b) Find all Nash equilibria of the game.

[7] Informally, conjecture is independent if the play of 1 does not provide any information about that play of 2

References

1. P. Battigalli, *Game Theory: Analysis of Strategic Thinking*, lecture notes, 2017.