GAME THEORY — SOLUTION SET 2

WINTER 2018 — PAULI MURTO, ANDREY ZHUKOV

In problem 1 below we analyze the strategic form of the alternating offer bargaining game, and discuss rationalizable and Nash Equilibrium (NE) strategies - with the view of having them as a reference point for comparison with subgame perfect equilibria (SPE) outcomes of the game which we will discuss in the lecture notes 3.

Problem 1 (Strategic form of the bargaining game)

We consider an alternating offer bargaining game à la Rubinstein (1982). Two players negotiate on how to split a pie of size 1. There is infinitely many stages-opportunities to agree on the split of the pie. The game begins with period 1, where player 1 suggests that she gets a share $x_1^1 \in [0,1]$ of the pie, and player 2 gets a share $1 - x_1^1$. Provided that the agreement has not been reached prior to period t, player 1 makes an offer $x_1^t \in [0,1]$ if t is odd, and player 2 makes an offer $x_1^t \in [0,1]$ if t is even. Future is discounted by a factor of δ: if players agree on the split x_1^t in period t, then the payoff of player 1 is $\delta^{t-1}x_1^t$, and the payoff of player 2 is $\delta^{t-1}(1 - x^{t-1})$. The payoff to never reaching an agreement is 0 for both players.

a) Give a formal definition of a strategy for players 1 and 2

Solution. Informally, a strategy of a player in a dynamic game is a history contingent plan of actions. Formally, let H_i be the set of information sets of player i in a game.[1] Then, a strategy for player i,s_i, is a mapping from the set of i's information sets into the set of probability distributions over actions available at information set $h_i \in H_i$. Following the lecture notes, let $A_i(h_i)$ be the set of actions available to player i at information set h_i. Then, the (behavioral) strategy is $s_i : H_i \to \Delta(A_i)$ such that $\mathrm{supp}(s_i(h_i)) \subset A_i(h_i)$. In words: a strategy specifies which action (mixed or pure) a player i takes given that his information set (history) is h_i with the restriction that the strategy must not prescribe actions which

[1] Information set is defined in lecture notes 1

are not available at the given information set. In the bargaining game described above, when it is time for player 1 to move in period t, the set of his period t information sets is $H_1^t := [0,1]^{t-1}$ - that is, the set of all possible offers players made up until period t, with the convention that $H^1 := \emptyset$. Then, the set of all information sets of player 1 in the game is $H := \cup_{t=1}^{\infty} H^t = \cup_{t=1}^{\infty}[0,1]^{t-1}$. Define an element of H^t by h^t, and then the (behavior) strategy is a map, $s_1 : \cup_{t=1}^{\infty}[0,1]^{t-1} \to \Delta(A_1)$, such that $supp(s_1(h^t)) \subset \{\text{accept}, \text{refuse}\}$ if t is even; and $supp(s_1(h^t)) \subset [0,1]$ if t is odd. That concludes defining a strategy for player 1. Definition of a strategy for player 2 is similar to that of player 1, and therefore is omitted here.

b) Consider the following strategy of player 2: "Regardless of the history of the game, refuse all offers but $x_1^t = 0$ in the odd periods, and always offer $x_1^t = 0$ in the even periods". Is this strategy rationalizable? Does your answer depend on δ?

Solution. Now, notice that h^t is a t-dimensional vector of offers made in the game prior to period t. Denote t^{th} element of the vector by $h^t(t)$.

Formally, the strategy for player 2 is

$$s_2'(h^t) = \begin{cases} 0 \text{ if } h^t \in H_2^t \text{ and } t \text{ is even,} \\ \text{refuse if } h^t \in H_2^t \text{ and } h^t(t) \neq 0 \text{ and } t \text{ is odd,} \\ \text{accept if } h^t \in H_2^t \text{ and } h^t(t) = 0 \text{ and } t \text{ is odd} \end{cases}$$

Consider the following strategy for player 1:

$$s_1'(h^t) = \begin{cases} 0 \text{ if } h^t \in H_1^t \text{ and } t \text{ is odd,} \\ \text{accept if } h^t \in H_1^t \text{ and } t \text{ is even} \end{cases}$$

Then, s_1' is a best response to s_2', s_2' is a best response to s_1', and hence strategy profile (s_1', s_2') constitutes NE of the game. Further, NE strategy is a best response, hence rationalizable.

c) Can you construct Nash Equilibrium where the agreement is reached in period 100 with any division of the pie?

Solution. Yes. The construction is similar to that of part *b)*, more details are in the class.

As opposed to NE, SPE imposes sequential rationality which makes it an attractive solution concept for multi-stage games. In problem 2 we compare NE predictions to SPE predictions of the important multi-stage game which was first analyzed in evolutionary biology to study a conflict where two players compete for an exclusive resource. In economics this game has been applied to study, among other things, *R&D* races and political lobbying.

Problem 2 (SPE in the war of attrition)

Two players are fighting for a prize whose current value at any time $t = 0, 1, 2, ...$ *is* $v > 1$*. Fighting costs* 1 *unit per period. The game ends as soon as one of the players stops fighting. If one player stops fighting in period* t*, he gets no prize and incurs no more costs, while his opponent wins the prize without incurring a fighting cost. If both players stop fighting at the same period, then neither of them gets the prize. The players discount their costs and payoffs with discount factor* δ *per period.*

This is a multi-stage game with observed actions, where the action set for each player in period t *is* $A_i(t) = \{0, 1\}$*, where* 0 *means continue fighting and* 1 *means stop. A pure strategy* s_i *is a mapping* $s_i : \{0, 1, ...\} \to A_i(t)$ *such that* $s_i(t)$ *descibes the action that a player takes in period* t *if no player has stopped the game in periods* $0, ..., t-1$*. A behavior strategy* $b_i(t)$ *defines a probability of stopping in period* t *if no player has yet stopped.*

a) Consider a strategy profile $s_1(t) = 1$ for all t and $s_2(t) = 0$ for all t. Is this a Nash equilibrium?

Solution. This is an equilibrium: given the behavior of player 2, player 1 has no incentive to fight. Player 2 gets utility v so he has no incentive to deviate.

b) Find a stationary symmetric Nash equilibrium, where both players stop with the same constant probability in each period.

Solution. (By stationary one means equilibria with strategies that are independent of t.) Let p be this probability of stopping. The condition for a mixed strategy equilibrium is that a player is indifferent between fighting and dropping out. In any period the utility from fighting in the present period is $pv + (1-p) \cdot (-1)$, since player 2 succumbs with probability p and fight with probability $1-p$. The continuation value (value of the future that arises after $(0,0)$) is zero. Players mix in the next period which implies that they are indifferent between fighting and stopping. Stopping gives a zero payoff, and hence the expected payoff after any action in the support of the mixed strategy is also zero. Therefore, we can ignore the continuation value. The utility from dropping out is 0. Thus the equilibrium condition is

$$pv + (1-p) \cdot (-1) = 0$$
$$p = \frac{1}{1+v}$$

c) Are the strategy profiles considered above subgame perfect equilibria?

Solution. This is because all stationary Nash equilibria are subgame perfect equilibria for stationary multistage games. In the game in question, previous fights are sunk cost and the time horizon in infinite, and hence all periods are equivalent to the first period. Therefore, the same argumentation, which was used for period 1 in a) and b), can be used for later periods as well. All stationary NE satisfy the one-step deviation condition.

d) Can you think of other strategy profiles that would constitute a subgame perfect equilibrium?

Solution. The equilibrium in (a) can obviously be reversed: where player 1 stops immediately and player 2 never stops: $s_1(t) = 0$ for all t and $s_2(t) = 1$ for all t. We could also combine profiles in a) and b). For example, the following is a SPE:

$$s_1 = (1, p, p, \dots)$$
$$s_2 = (0, p, p, \dots).$$

There is also a mixed strategy equilibrium, where players stop every second period with probability ρ, i.e. their strategies assign probabilities

$(0, \rho, 0, \rho, 0, \ldots)$ and $(\rho, 0, \rho, 0, \ldots)$ to quitting. The argument why this works is similar to the symmetric equilibrium. The important condition is that the player who is mixing between stopping and continuing must be indifferent (the value of the game is zero for her).

The player who is not mixing has a value: $\rho v + (1-\rho)(-1)$. (The not mixing player will mix in the following period, and hence her continuation value is zero.) The player, who is mixing now, has a continuation value of $\delta(\rho v + (1-\rho)(-1))$. Her indifference condition yields:

$$\delta(\rho v + (1-\rho)(-1)) - 1 = 0$$
$$\Leftrightarrow \rho = \frac{1+\delta}{\delta(1+v)}.$$

Can you see why there cannot be a period in which both players fight with probability one?

SPE is our “default" solution concept for multi-stage games, therefore it is important to carefully and critically examine it. In problem 3 we compare NE and SPE outcomes of the simple games with their actual play in experimental setting [based on Goeree and Holt (2001)].

Problem 3 (Experimental evidence on SPE)

a) Consider the extensive form of the game in Figure 1. What are NE and SPE of the game? In the experimental setting, 16% of randomly matched pairs played the game with the outcome of $(80, 50)$, and the rest played the game with the outcome of $(90, 70)$. How do you interpret this finding, as in what does it tell us about SPE as a solution concept?

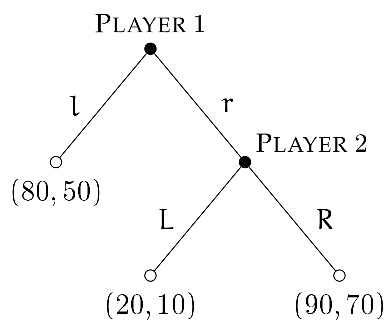

Figure 1: Should you trust others to be rational?

Solution. Strategic form of the game in Figure 1 is presented in Table 1. (l, L) and (r, R) are pure strategy NE. Further, denote probability that player 1 chooses l by σ_1, and probability that 2 chooses r by σ_2. Next, notice that if there exists a mixed-strategy equilibrium of the game, it must be the case that in such equilibrium $\sigma_1 = 1$, since this is the only way player 2 could be indifferent between L and R. Therefore, there exists a continuum of mixed-strategy NE $\{(\sigma_1, \sigma_2) : \sigma_1 = 1, \sigma_2 \geqslant \frac{1}{7}\}$.

	L	R
l	80, 50	80, 50
r	20, 10	90, 70

Table 1: Strategic form of the game in Figure 1

The only SPE of the game is (r, R). I would interpret the finding as supporting the refinement of NE suggested by SPE.

b) Consider the extensive form of the game in Figure 2. What are NE and SPE of the game? How do you think empirical distribution of outcomes changes compared to the game in *a)* and why?

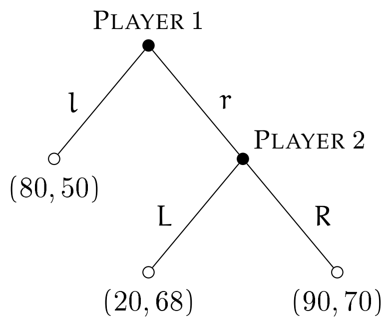

Figure 2: Revisited "Should you trust others to be rational?"

Solution. Strategic form of the game in Figure 2 is presented in Table 2. (l, L) and (r, R) are pure strategy NE. Further, denote probability that player 1 chooses l by σ_1, and probability that 2 chooses r by σ_2. Next, notice that if there exists a mixed-strategy equilibrium of the game, it must be the case that in such equilibrium $\sigma_1 = 1$, since this is the only way player 2 could be indifferent between L and R. Therefore, there exists a continuum of mixed-strategy NE $\{(\sigma_1, \sigma_2) : \sigma_1 = 1, \sigma_2 \geqslant \frac{1}{7}\}$. The only SPE of the game is (r, R).

	L	R
l	80, 50	80, 50
r	20, 68	90, 70

Table 2: Strategic form of the game in Figure 2

Informally, I would think that here player 1 was more likely to play l, since as opposed to the game in part a), player 2 does not have "clear" preference for playing R, and therefore player 1 might not trust that player 2 is attentive/rational enough to play R. This corresponds to the finding in Goeree and Holt (2001) - 52% of the randomly matched pairs played the game with the outcome $(70, 60)$ [reflecting the idea that player 1 does not trust player 2 to be "rational"]; 12% of the games ended with the outcome $(20, 68)$ [confirming that player 1 is correct not to trust in "rationality" of player 2]; and 36% played $(90, 70)$.

Goeree and Holt (2001) try to spin the result as exhibiting the flaws of SPE, claiming that sometimes SPE might be refining too "rigorously" . My opinion is that the individuals in the lab played a game which is different from the one depicted in Figure 2 - in particular, players in the lab could not really know payoff functions of their opponents and therefore could not exclude that there were some "evil" -type opponents in the game, who cared about minimizing other players' monetary payments. But then, of course, they are playing a different game than the one depicted in Figure 2, and whatever happened in the lab should not tell us much about what the outcomes of the experiment would have been if players really partook in the correct game.

c) Consider the extensive form of the game in Figure 3. What are NE and SPE of the game? In the experimental setting, 12% of randomly matched pairs played the game with the outcome $(70, 60)$, and the rest played the game with the outcome of $(90, 50)$. How do you interpret this finding?

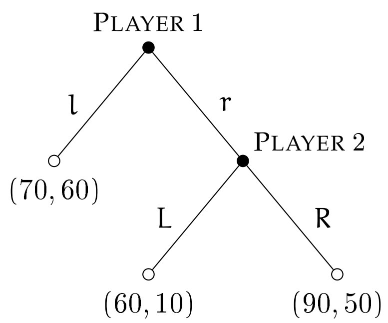

Figure 3: Should you believe a threat which is not credible?

Solution. Strategic form of the game in Figure 3 is presented in Table 3. (l, L) and (r, R) are pure strategy NE. Further, denote probability that player 1 chooses l by σ_1, and probability that 2 chooses r by σ_2. Next, notice that if there exists a mixed-strategy equilibrium of the game, it must be the case that in such equilibrium $\sigma_1 = 1$, since this is the only way player 2 could be indifferent between L and R. Therefore, there exists a continuum of mixed-strategy NE $\{(\sigma_1, \sigma_2) : \sigma_1 = 1, \sigma_2 \geqslant \frac{2}{3}\}$. The only SPE of the game is (r, R).

	L	R
l	70, 60	70, 60
r	60, 10	90, 50

Table 3: Strategic form of the game in Figure 3

(l, L) relies on the not credible threat of player 2 to go L when it is his turn to move.

d) Consider the extensive form of the game in Figure 4. What are NE and SPE of the game? In the experimental setting, 32% of randomly matched pairs played the game with the outcome $(70, 60)$, 32% played the game with the outcome of $(60, 48)$ and the rest played the game with the outcome of $(90, 50)$. Does this empirical distribution support SPE? Why?

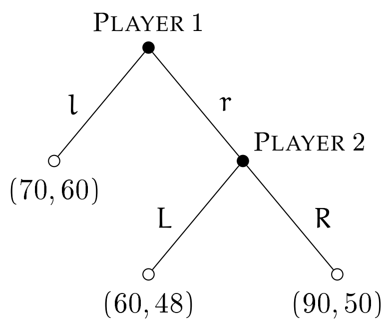

Figure 4: Revisited "Should you believe a threat which is not credible?"

Solution. Strategic form of the game in Figure 4 is presented in Table 4. (l, L) and (r, R) are pure strategy NE. Further, denote probability that player 1 chooses l by σ_1, and probability that 2 chooses r by σ_2. Next, notice that if there exists a mixed-strategy equilibrium of the game, it must be the case that in such equilibrium $\sigma_1 = 1$, since this is the only way player 2 could be indifferent between L and R. Therefore, there exists a continuum of mixed-strategy NE $\{(\sigma_1, \sigma_2) : \sigma_1 = 1, \sigma_2 \geqslant \frac{2}{3}\}$. The only SPE of the game is (r, R).

	L	R
l	70, 60	70, 60
r	60, 10	90, 50

Table 4: Strategic form of the game in Figure 4

Problem 4 below demonstrates issues with using SPE for analysis of dynamic games with incomplete information.

Problem 4 (SPE in games with incomplete information)

a) What are SPE of the game in Figure 5? Are all of them sequentially rational?

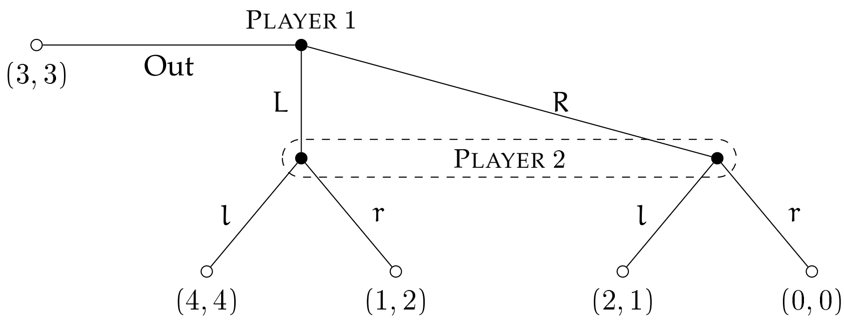

Figure 5: Deadlock game with an outside option

Solution. The game in Figure 5 has no proper subgames, and therefore the sets of NE and SPE coincide. To find NE of the game, we represent the game in the strategic form in Table 5.

	l	r
Out	3,3	3,3
L	4,4	1,2
R	2,1	0,0

Table 5: Strategic form of the game in Figure 5

Pure strategy SPE of the game are (Out, r) and (L, l). Further, similar to problem 1, there is a continuum of mixed-strategy equilibria.

b) In the game of Figure 6, Nature chooses L with probability $\frac{3}{4}$ and R with probability $\frac{1}{4}$. What are SPE of the game?

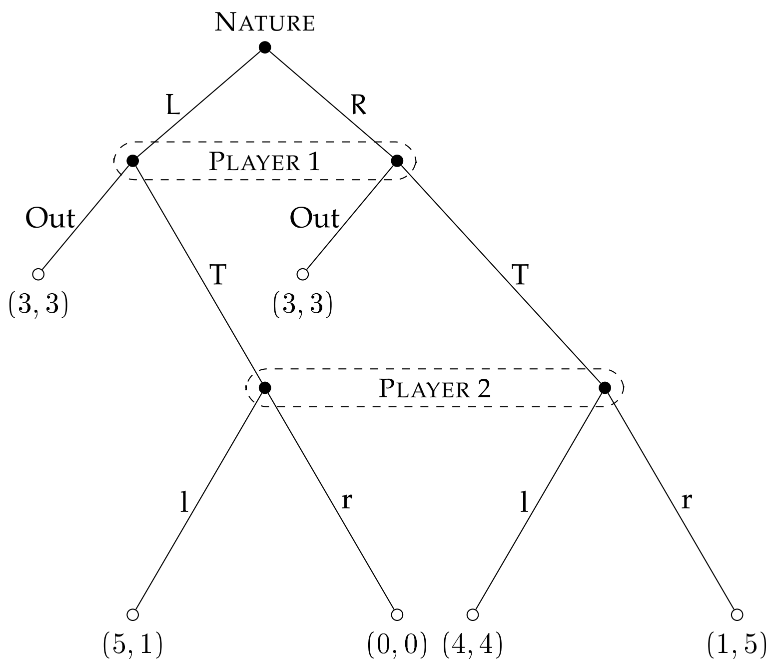

Figure 6: SPE supported by inconsistent beliefs

Solution. The game in Figure 6 has no proper subgames, and therefore the sets of NE and SPE coincide. To find NE of the game, we represent the game in the strategic form in Table 6.

	l	r
Out	3,3	3,3
T	$\frac{19}{4}, \frac{7}{4}$	$\frac{1}{4}, \frac{5}{4}$

Table 6: Strategic form of the game in Figure 6

Two pure-strategy SPE are $(\mathsf{Out}, \mathsf{r})$ and (T, l).

Problem 5 (Perfect Bayesian Equilibrium) *What are Perfect Bayesian Equilibria of the game in problem* 4*?*

Solution. In the game of part a), the unique PBE is (L, l) since regardless of the belief system, player 2 will play l when he finds himself at the information set where his decision is needed, and as opposed to SPE - PBE forces us to define the belief system. In the game of part b), all the SPE are also PBE since PBE gives us freedom to choose beliefs at the information sets which are reached with probability 0 in equilibrium. Hence the need for the sequential equilibrium, which we will discuss in the next problem session.

References

1. Jacob K. Goeree, Charles A. Holt. “The little treasures of game theory and ten intuitive contradictions". AER, 2001.