We ended the last problem set with the game which exposed the drawbacks of PBE in the games of incomplete information. Problem 1 below presents some games where PBE does not perform well, and introduces sequential equilibrium (SE) as a remedy to the exposed problems.

Problem 1 (Sequential equilibrium)

a) In the game of Figure 1, Nature chooses L with probability $\frac{3}{4}$. What are SE of the game? Compare them to PBE found in problem set 2: does SE make more appealing predictions on the outcome of the game?

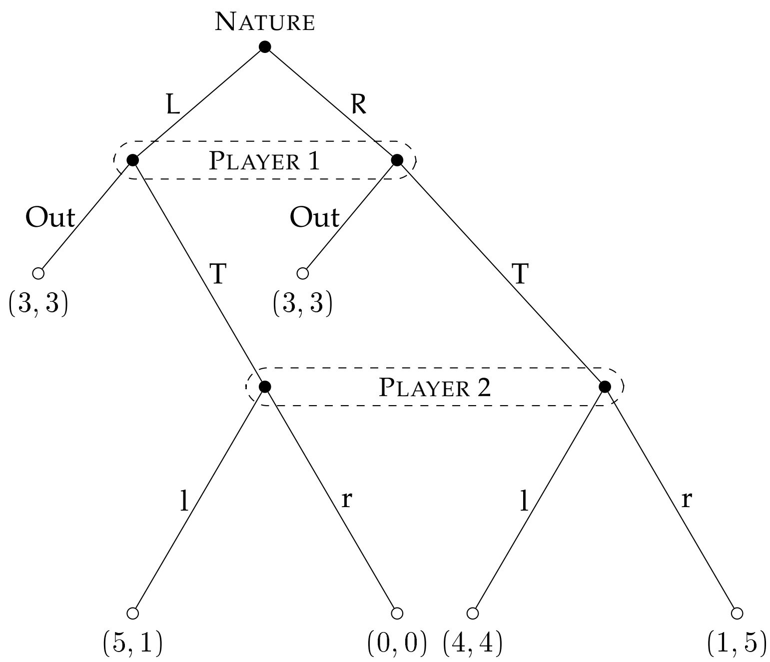


Figure 1: SPE supported by inconsistent beliefs

Solution. Recall a definition of SE from the lecture notes: a pair of behavioral strategy and a belief system (b, μ) is sequential equilibrium if two conditions are satisfied:

1. b is sequentially rational given μ
2. *consistency:* there exists a sequence $(b^n, \mu^n) \to (b, \mu)$ such that for all n, b^n puts a positive probability on all available actions, and for any information set h and any $x \in h$, $\mu_h^n(x) = \frac{P^{b^n}(x)}{P^{b^n}(h)}$

Next notice that the only consistent belief for player 2 is $\mu_2(\{(L,T),(R,T)\})(R,T) = \frac{1}{4}$, where $\mu_2(\{(L,T),(R,T)\})(L,T)$ is the belief assigned to the path (L,T) at the information set at which player 2 makes a decision, and which could be reached through either a path (L,T), or a path (R,T). To see this, notice that for any behavioral strategy of player 1 which assigns positive probability on all available actions, $b_1^n(Out) > 0$ and $b_1^n(T) > 0$, it is the case that $\mu_2^n(\{(L,T),(R,T)\})(L,T) = \frac{\frac{3}{4}b_1^n(T)}{\frac{3}{4}b_1^n(T)+\frac{1}{4}b_1^n(T)} = \frac{3}{4}$. Since the only consistent belief is $\mu_2^n(\{(L,T),(R,T)\})(L,T) = \frac{3}{4}$, the only behavioral strategy of player 2 which is sequentially rational given μ_2 is $b_2(l) = 1$. Then, proceeding backwards, the only sequentially rational behavioral strategy of player 1 is $b_2(T) = 1$.

Thus, the game has unique SE, (b, μ). Let $b = (b_1, b_2)$ be behavioral strategies of players 1 and 2 respectively, and $\mu = (\mu_1, \mu_2)$ be the beliefs at information sets at which players 1 and 2 make decisions respectively. Then the unique SE is $b = (b_1(Out) = 1, b_2(l) = 1)$, $\mu = (\mu_1(\{(L),(R)\})(L) = \frac{3}{4}, \mu_2\{(L,T),(R,T)\}(L,T) = \frac{3}{4})$.

b) Show that the set of SPE is a proper subset of the PBE in the game of Figure 2. What are SE of the game?

Solution. There is unique SPE in the game: Player 1 plays In in his first information set, R in his second information set, and player 2 goes r when it is his turn to move.

Let $b = (b_1, b_2, b_3)$ be behavioral strategies of the players, where b_1 is probability player 1 plays Out in his first information set, b_2 is probability player 1 plays L in his second information set, and b_3 is probability that player 2 plays l when it is his turn to move. Further, let μ be a belief system, where μ is the belief assigned to the path (In, L) at the information set

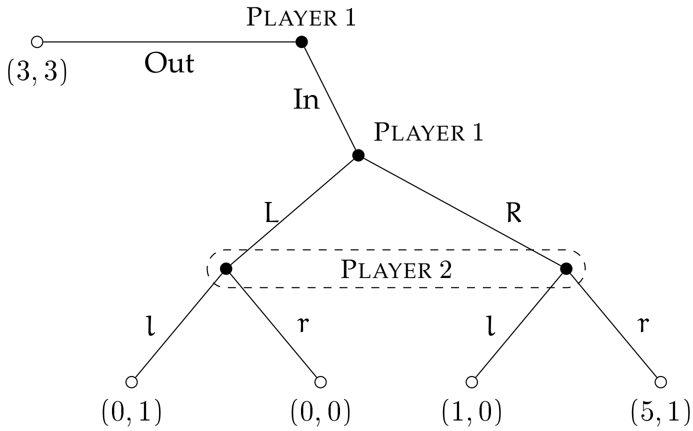


Figure 2: Pathological PBE

at which player 2 makes a decision, and which could be reached through either a path (In, L), or a path (In, R).

Then $b = (b_1 = 1, b_2 = 0, b_3 = 1)$, $\mu = 1$ is PBE: b is sequentially rational given μ, and μ is derived from b using Bayes rule whenever possible. The other PBE coincides with SPE when $\mu = 0$.

It is easy to verify that with $b = (b_1 = 1, b_2 = 0, b_3 = 1)$, belief $\mu = 1$ is inconsistent - there does not exist a sequence of $(b^n, \mu^n) \to (b, \mu)$ where $\mu^n = b_2^n$, since if $b_2^n \to 0$ then $\mu^n \to 0$.

In lecture notes 3, we analyzed SPE of sequential bargaining game with alternating offer protocol - i.e., players took turn making offers. In problem 2, we generalize the protocol but focus on a two-period model - and show that the main insights of the model do not change.

Problem 2 (Random proposer protocol) *Players* 1 *and* 2 *want to divide a dollar and they have two periods to reach an agreement. Players are risk-neutral, and if the agreement is not reached by the end of period* 2*, Nature sets the dollar on fire. Nature chooses player* 1 *to make a proposal on a division of the dollar in period* $t \in \{1, 2\}$ *with probability* π*, and with complementary probability it is player* 2*, who gets to make a proposal in period* t*. That is, in period* 1 *the player recognized as a proposer by Nature suggests a division of the dollar* $(x^1, 1-x^1)$*, and the other player can either accept or refuse this proposal. If the offer is accepted, the game ends with payoffs* $(x^1, 1 - x^1)$*. If the offer is refused, the game moves to period* 2*, where Nature chooses a proposer again [she chooses player* 1 *with probability* π*,* *and player 2 with probability $(1-\pi)$], and the recognized player proposes a division $(x^2, 1-x^2)$. If the offer is accepted, the game ends with payoffs $(\delta x^2, \delta(1-x^2))$. If the offer is rejected, the game ends with payoffs $(0, 0)$. What is the unique* SPE *of the game?*

Solution

The game is solved by backward induction. If player 1 is chosen to make a proposal in period 2, then he offers $(1, 0)$, which is accepted by player 2. If player 2 is chosen to make a proposal in period 2, then she offers $(0, 1)$, which is accepted by player 1. Going backwards, in period 1, player 1 would never accept an offer of $x_1^1 < \delta\pi$, since she gets expected payoff of $\delta\pi$ by refusing an offer. Similarly, player 2 never accepts an offer $1 - x_1^1 < \delta(1-\pi)$, since she gets expected payoff of $\delta(1-\pi)$ by refusing an offer. Next, in period 1 player 1 never refuses an offer of $x_1^1 \geqslant \delta\pi$, and 2 never refuses an offer $1 - x_1^1 \geqslant \delta(1-\pi)$. Therefore, if player 1 makes an offer in period 1, she offers $x_1^1 = 1 - \delta(1-\pi)$, which is accepted by player 2. If it is player 2 who makes an offer in period 2, she offers $x_1^1 = \delta\pi$ which is accepted by player 1. In any case, there is immediate agreement with the expected payoffs $(\pi, 1-\pi)$.

Problem 3 *A popular strategy suggestion for playing a repeated prisoner's dilemma is called tit-for-tat. In that strategy, both players start by cooperating (C, C) and in any period t, they replicate the action of their opponent in period $t-1$. Consider the infinitely repeated game where both players discount future with discount factor $\delta < 1$. The stage-game payoffs are:*

	C	D
C	$3, 3$	$0, 4$
D	$4, 0$	$1, 1$

Write down a formal definition for the tit-for-tat strategy. Is the strategy profile where both players play tit-for-tat a Nash equilibrium?
Is it a sub-game perfect Nash equilibrium?

Solution.

A tit-for-tat strategy for player i is defined as

$$\hat{s_i}(h^t) = \begin{cases} C & \text{if } t = 0 \\ C & \text{if } a_j^{t-1} = C,\ j \neq i \\ D & \text{otherwise.} \end{cases}$$

Player i's objective function is to maximize the normalized sum

$$U_i(s) = (1-\delta) \sum_{t=0}^{\infty} \delta^t u_i^t \left(s_i^t(h^t), s_{-i}^t(h^t) \right).$$

Notice that in prisoner's dilemma, the unique stage game Nash equilibrium yields both players their minmax payoff of 1.

Both players playing tit-for-tat is a Nash equilibrium for high enough δ. If both players always cooperate, they get normalized payoffs of 3. We cannot use OSDP for testing a NE since it is essentially backward induction principle and hence restricted to testing a SPE. Therefore, we have to think what would be the best deviation among all kind of deviations. Let's take it given that player 2 follows tit-fot-tat. Note first that if it is optimal for player 1 to play D in the first period, it is optimal to play D whenever player 2 plays C, which happens iff player 1 has played C in the previous period. Now we see that if (D, C) is optimal behavior in the first two periods, optimal strategy is to continue $(D, C, D, C \dots)$ forever. Similarly, if (D, D) is optimal for the first two periods, it is optimal to continue similarly: $(D, D, D, \dots)$. We can restrict our attention to these deviations.

$$u_1(DC\dots, \hat{s_2}) = (1-\delta)\left[4(1+\delta^2+\delta^4+\dots)+0\right] = \frac{4}{1+\delta}.$$
$$u_1(DD\dots, \hat{s_2}) = (1-\delta)4+\delta = 4-3\delta.$$

Deviations are unprofitable when

$$3 \geqslant \frac{4}{1+\delta} \Leftrightarrow \delta \geqslant \frac{1}{3}.$$
$$3 \geqslant 4-3\delta \Leftrightarrow \delta \geqslant \frac{1}{3}.$$

Is tit-for-tat strategy a subgame perfect equilibrium? For tit-for-tat strategy profile to constitute a subgame perfect equilibrium, no player should have an incentive to deviate in any of the possible subgames that could occur along any path of play if both players play according to their equilibrium strategies. There are four types of subgames to consider, depending what happened in the previous period. Let's consider each of them at a time, using the one-step deviation principle:

1. The last realization was (C, C). If player 1 follows tit-for-tat, thus continuing with C, his payoff is given by

$$(1-\delta)\left[3\left(1+\delta+\delta^2+...\right)\right]=3.$$

If player 1 deviates, the sequence of outcomes is $(D, C), (C, D), (D, C), (C, D), ...$, and his payoff will be

$$(1-\delta)\left[4\left(1+\delta^2+\delta^4+...\right)\right]=\frac{4}{1+\delta}.$$

Deviation is not profitable when $\delta \geqslant \frac{1}{3}$.

2. The last realization was (C, D). If player 1 follows tit-for-tat, the resulting sequence of outcomes will be $(D, C), (C, D), (D, C), ...$, to which the payoff is $\frac{4}{1+\delta}$. If player 1 deviates and cooperates, the sequence of outcomes will be $(C, C), (C, C), (C, C), ...$, to which the payoff is 3. Deviating is not profitable as long as

$$\begin{aligned} \frac{4}{1+\delta} &\geqslant 3 \\ \Leftrightarrow \delta &\leqslant \frac{1}{3}. \end{aligned}$$

3. The last realization was (D, C). If player 1 follows tit-for-tat, the resulting sequence of outcomes will be $(C, D), (D, C), (C, D), ...$, to which the payoff is given by

$$\begin{aligned} &(1-\delta)\left[0(1+\delta^2+\delta^4+...)+4\left(\delta+\delta^3+\delta^5+...\right)\right] \\ &\qquad =\frac{4\delta}{1+\delta}. \end{aligned}$$

If player 1 deviates and plays D instead, the sequence of outcomes will be $(D, D), (D, D), (D, D), \ldots$, to which the payoff would be 1. Deviation is not profitable when $\frac{4\delta}{1+\delta} \geqslant 1 \Leftrightarrow \delta \geqslant \frac{1}{3}$.

4. The last realization was (D,D). Thus the sequence of play will be (D,D),(D,D),(D,D)..., which will result in a payoff of 1. If player 1 deviates to C instead, it will result in a sequence of play (C,D),(D,C),(C,D)... And the payoff will be

$$(1-\delta)\left[0(1+\delta^2+\delta^4+\ldots)+4\left(\delta+\delta^3+\delta^5+\ldots\right)\right] \\ = \frac{4\delta}{1+\delta}.$$

Deviation is not profitable as long as $\frac{4\delta}{1+\delta} \leqslant 1 \Leftrightarrow \delta \leqslant \frac{1}{3}$.

So, tit for tat is a subgame perfect equilibrium if and only if $\delta = (1/3)$.

Problem 4 *Consider a two-stage game with observed actions, where in the first stage players choose simultaneously U1 or D1 (player 1) and L1 or R1 (player 2), and in the second stage players choose simultaneously U2 or D2 (player 1) and L2 or R2 (player 2). The payoffs of the stage games are shown in the tables below:*

First stage:

	L1	R1
U1	$2, 2$	$-1, 3$
D1	$3, -1$	$0, 0$

Second stage:

	L2	R2
U2	$6, 4$	$3, 3$
D2	$3, 3$	$4, 6$

The players maximize the sum of their stage-game payoffs.

a) Find the subgame-perfect equilibria of this game.

Solution. In the first stage game there is only one Nash equilibrium, (D1,R1), since D1 and R1 are strictly dominant actions for the players. In the second stage game, there are three Nash equilibria: (U2,L2), (D2,R2), ((3/4,1/4),(1/4,3/4)). Like we have seen previously, any combination of these Nash equilibria are a subgame perfect equilibrium, i.e. player 1 playing $s_1^1 = D1, s_1^2(h^2) = U2\ \forall h^2$ and player 2 playing $s_2^1 = R1, s_2^2(h^2) = L2\ \forall h^2$ is a subgame perfect equilibrium.

Note that we cannot support $(U1, L1)$ as the first stage outcome with any punishment strategy in the second stage game, since deviating yields a payoff of 1 and the mixed strategy equilibrium still gives a payoff of (3.75), which is only 0.25 lower than what the player who is getting a lower payoff either from (U2,L2), (D2,R2) is getting. Thus it is profitable for that player to deviate. We can support (U1,R1) or (D1,L1) though by playing the Nash equilibrium in the second stage that gives the higher payoff to the player that gets -1 in the first stage. So strategies defined as

$$s_1(h^1) = D1, s_1(h^2) = \begin{cases} \text{D2 if } h^2 = (D1, L1) \\ \text{U2 otherwise} \end{cases}$$

$$s_2(h^1) = L1, s_2(h^2) = \begin{cases} \text{R2 if } h^2 = (D1, L1) \\ \text{L2 otherwise} \end{cases}$$

Constitute a SPE with payoffs (7,5). Naturally, the strategies where player 1's and 2's roles are reversed is also a SPE.

b) Suppose that the players can jointly observe the outcome y_1 of a public randomizing device before choosing their first-stage actions, where y_1 is drawn from uniform distribution on the unit interval. Find the set of subgame-perfect equilibria, and compare the set of possible payoffs against the possible payoffs in a).

Solution. The public randomization device allows players to attain all payoffs contained in the convex combinations of the previous SPE. That is the players can now condition their play on y_1 and play different SPE depending on the value of y_1. Thus all payoffs that are convex combinations of $(3.75, 3.75)$, $(6, 4)$, $(7, 5)$, $(5, 7)$ and $(4, 6)$ are now attainable.

c) Suppose that the players jointly observe y_1 at the beginning of stage 1 and y_2 at the beginning of stage 2, where y_1 and y_2 are independent draws from a uniform distribution on a unit interval. Again, find the sub-game perfect equilibriua and possible payoffs.

Solution. With y_2 we can also attain all the convex combinations of the second stage game payoffs separately, i.e. choosing one of (3.75,3.75), (6,4) or (4,6) is possible conditional on y_2. This allows us to support (U1,L1) as a first stage outcome. To see why, let's consider the following strategies:

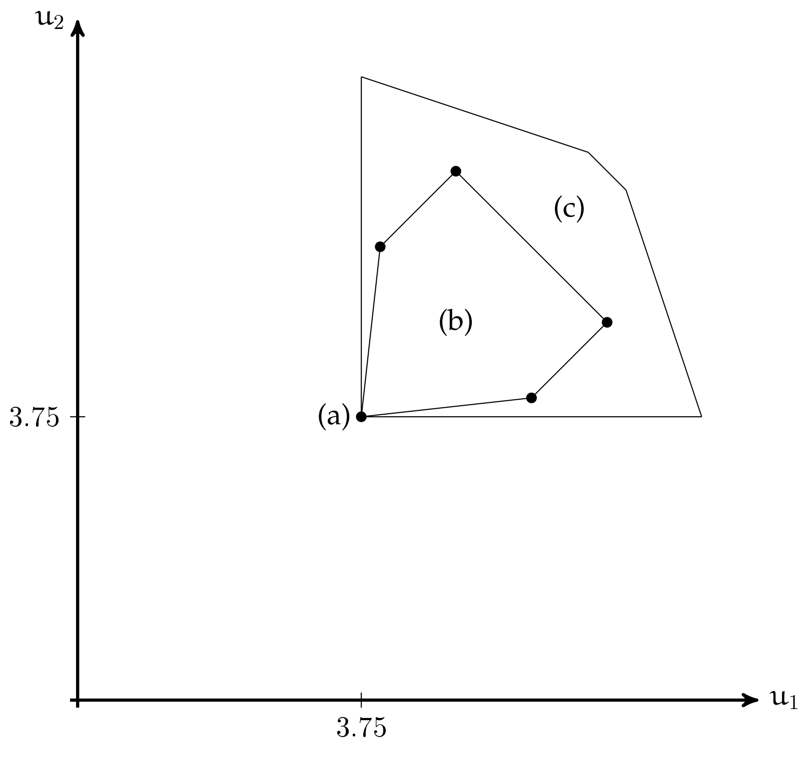
$$s_1(h^1) = U1, s_1(h^2) = \begin{cases} D2 \text{ if } h^2 = (U1, L1) \text{ and } y_2 < \frac{1}{2} \\ U2 \text{ if } h^2 = (U1, L1) \text{ and } y_2 \geqslant \frac{1}{2} \\ (\frac{3}{4}, \frac{1}{4}) \text{ otherwise} \end{cases}$$

$$s_2(h^1) = L1, s_2(h^2) = \begin{cases} R2 \text{ if } h^2 = (U1, L1) \text{ and } y_2 < \frac{1}{2} \\ L2 \text{ if } h^2 = (U1, L1) \text{ and } y_2 \geqslant \frac{1}{2} \\ (\frac{1}{4}, \frac{3}{4}) \text{ otherwise} \end{cases}$$

The expected payoff from second stage is now 5. By deviating the players can gain 1, but will lose 1.25 in the second stage, so (U1,L1) is supportable.

One can support similarly any not stage NE equilibrium, which gives the player, who would have incentives to deviate in a one-period game, a payoff of at least 4.75 in the second period. Thus the set of attainable payoffs is now extended to the convex hull of $(3.75, 3.75)$, $(3.75, 8.25)$, $(6.75, 7.25)$, $(7.25, 6.75)$, and $(8.25, 3.75)$.

Following figure presents the SPE sets in parts (a), (b), and (c).



Problem 5 (Folk Theorem) *Consider an infinitely repeated game with a stage game given in the following matrix:*

	L	R
U	$5, 0$	$0, 1$
M	$3, 0$	$3, 3$
D	$0, -1$	$0, -1$

Players have a common discount factor.

a) Find the minmax payoffs for each of the players.

Solution. The minmax payoff are: $\hat{v}_1 = 3, \hat{v}_2 = -1$.

b) Characterize the set of feasible payoff vectors of the stage game (Assume that a public randomization device is available).

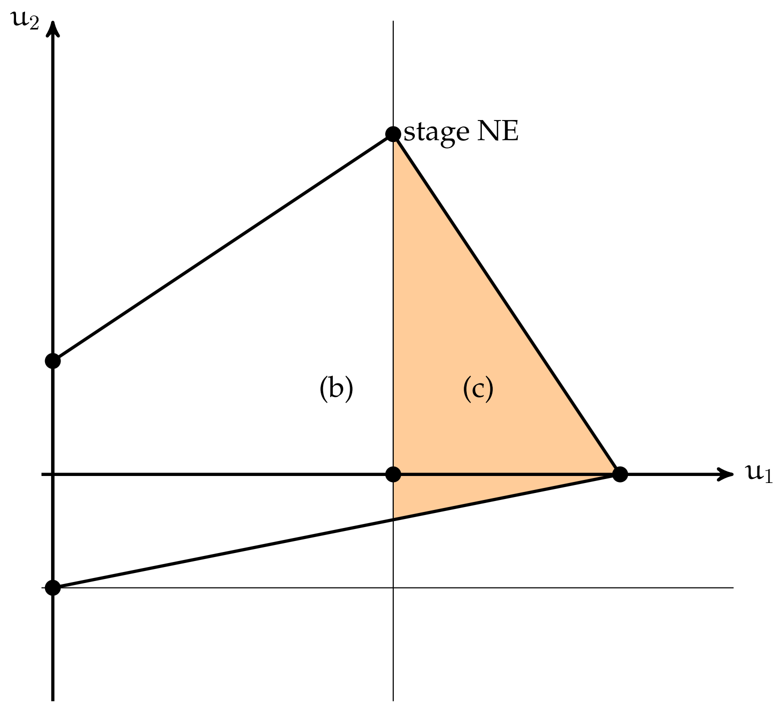
Solution. A public randomization device enables all payoffs that are convex combinations of some pure strategy payoffs (dots in the figure). The set is drawn in the figure on the next page.

c) What is the set of normalized payoff vectors for the repeated game, such that each element in the set is a subgame perfect equilibrium payoff vector for some value of the discount factor?

Solution. Any feasible and strictly individually rational payoff is a normalized SPE payoff for an infinitely repeated game with δ high enough. In addition, repeated play of a stage NE is always a SPE. This is equivalent as having:

$$\{SPE\} = \{(u_1, u_2) | (u_1, u_2) \in co\{(0, -1), (0, 1), (3, 3), (3, 0), (5, 0)\}, u_1 > 3, u_2 > -1\} \cup (3, 3).$$

Following figure illustrates this set.



d) Can you construct some subgame perfect equilibrium strategies leading to the constant play of (U, L) in the equilibrium path?

Solution. Strategies should punish players from deviations, including deviations from punishing the other player. Punishments has to be sequentially rational (NEa for subgames starting from a deviation). A natural punishment for player 1 is that player 2 always plays R. This punishment is automatically sequentially rational since it leads to a continuous play of the stage NE.

Punishment path for player 2 is trickier: constant play of D is not part of a NE. There has to be incentives for player 1 to do the punishment. In other words, the play should change back to (U, L) in the future. Following startegies constitute a SPE with δ high enough:

$$\hat{s_1}(h^t) = \begin{cases} U \text{ if } t = 0 \text{ or } a_2^{t-i} = L \quad \forall i \in \{1, \dots, T\} \text{ and player 1 has not deviated} \\ M \text{ if player 1 has deviated} \\ D \text{ if } a_2^{t-i} = R \text{ for some } i \in \{1, \dots, T\} \text{ and player 1 has not deviated} \end{cases}$$

$$\hat{s_2}(h^t) = \begin{cases} L \text{ if player 1 has not deviated} \\ R \text{ if player 1 has deviated.} \end{cases}$$

There are three different kind of states: the equilibrium path, the punishment path for player 1, and the punishment path for player 2 (there are actually T different states on this path). We have to check that no-one has a profitable one-step deviation in any of these states.

Player 1:

1. On the equilibrium path. Clearly, not.

2. On the punishment path for player 1. No, it is a stage NE.

3. On the punishment path for player 2. Deviation is not profitable iff

$$\delta^{T+1}5 \geqslant (1-\delta)5 + \delta 3 \Leftrightarrow \delta^{T+1} \geqslant 1 - \frac{2}{5}\delta. \tag{1}$$

Player 2:

1. On the equilibrium path. Deviation is not profitable iff

$$0 \geqslant (1-\delta)1 - (\delta - \delta^{T+1}) \Leftrightarrow \delta^{T+1} \leqslant 2\delta - 1. \tag{2}$$

2. On the punishment path for player 1. No, it is a stage NE.

3. On the punishment path for player 2. No, there are no short term or long term gains.

By combining 1 and 2, we get $1 - \frac{2}{5}\delta \leqslant \delta^{T+1} \leqslant 2\delta - 1$. For $T = 2$, this is satisfied e.g. for all $\delta > 0.9$.

e) Let's change the game so that payoffs for (D, L) and (D, R) are $(0, 0)$. Can there now be an equilibrium with a constant play of (U, L)?

Solution. No, there cannot be a punishment for player 2 since she already gets her minmax payoff. This would be the case even if the payoff structure was $u(D, L) = (0, 0)$ and $u(D, R) = (0, -1)$ so that there was a lower feasible payoff but the minmax payoff was still 0.

Problem 6 *Consider a model where two sellers sell an identical good to a single consumer (without storage possibilities) over an infinite horizon. The firms compete by setting prices simultaneously at the beginning of each period and the* *consumer chooses which of the prices to accept at the end of the stage. The consumer has unit demand in each period, i.e. she is willing to pay up to v in each period to buy one unit. Additional units are worthless to the buyer. Assume that the good can be produced at marginal cost* c.

a) Suppose that the buyer is myopic, i.e. she has a discount factor $\delta^C = 0$ whereas the firms are patient and have a discount factor $0 < \delta^F < 1$. What is the smallest δ^F that is compatible with collusive pricing in the market in subgame perfect equilibrium? I.e. for what δ^F is it possible to set prices $p_{it} = v$ for all i and all t *on the equilibrium* path? What is the punishment path supporting this? (Hint: what are the strategies of the players?)

Solution. Let's restrict to cases where the sellers don't set prices above v. The buyer accepts at least one price every period. If the buyer accepts both prices (when $p_{it} = p_{jt}$), both firms have an equal probability of getting their good sold in that period. Since the buyer is myopic and maximizes her periodic utility only, she does not condition her decision of acceptance on the publicly observed history of prices. Therefore, let buyer's strategy be given by

$$s_B^t(p_{it}) = \begin{cases} \text{Accept} & \text{if } p_{it} \leqslant p_{jt} \\ \text{Reject} & \text{otherwise} \end{cases}, \quad i \in \{1, 2\}, i \neq j.$$

Comment. A strategy for firm i is a function $s_i^t : H^t \to \mathbb{R}_+$ that assigns a price to every possible history in the game. Define the strategy of the buyer as $s_B^t : \mathbb{R}_+ \to \{accept, reject\}$ which determines for each element of the price vector $p = (p_{it}, p_{jt})$ that she observes whether to accept or reject it. The buyer accepts at least one price every period. If the buyer accepts both prices (when $p_{it} = p_{jt}$), both firms have an equal probability of getting their good sold in that period such that firm i's expected payoff from that period is $\frac{1}{2}(p_{it} - c)$, $i = 1, 2$. The decision that a consumer faces every period concerns the choice of the price that she pays in period t.

To find the smallest δ^F compatible with collusive pricing, the firms must use the most severe punishment available so that even the most impatient firm has no incentive to deviate. The minmax payoffs of both firms are

0, obtained when the price equals marginal cost c. Consider therefore the following grim-trigger strategy profile of the firms as a candidate for a SPE:

$$s_i^t(h^t) = \begin{cases} \nu & \text{if } t = 0 \\ \nu & \text{if } a^\tau = (\nu, \nu)\ \forall \tau < t \\ c & \text{otherwise} . \end{cases}$$

For this strategy profile to be subgame perfect, there must be no profitable deviations starting from any possible subgame. Suppose that in period $t-1$ collusion was sustained and the vector of quoted prices was (ν, ν). If firm i sticks to the equilibrium strategy from period t onwards, its payoff is given by $\frac{1}{2}(\nu - c)$. If it instead sets a price $p_{it} = \nu - \epsilon$, thereby selling its good with certainty in period t, and confines to the equilibrium strategy profile thereafter, it obtains a payoff of $(1-\delta)\left[(\nu - \epsilon - c) + 0\right]$. Deviation is not profitable as long as $\delta \geqslant \frac{1}{2}$.

b) Suppose next that all players, i.e. the sellers as well as the buyer have the same discount factor δ. Can you find an equilibrium where collusion is possible at a δ below that found in the previous part? (Hint: try to constract strategies for the sellers that reward the buyer for not falling for a price cut of the competitor)

Solution. What the firms are now able to do, once the consumer is patient, is to condition the consumer's future payoffs on her actions today. They can 'punish' the consumer if she falls for the price-cut and in turn 'reward' her for ignoring the price-cut and buying at the monopoly price instead. This in turn will in equilibrium make it unprofitable for firms to break the collusion.

Consider the following strategies for the players.

Consumer: Buy at price $p_t = \nu$ as long as both firms quote it. If a unilateral price-cut of at most ϵ^* is observed in any period t, ignore it and buy at the monopoly price $p_t = \nu$ and thereafter buy at the price of $p_\tau = c$ in all periods $\tau > t$. If a unilateral price-cut of more than ϵ^* is observed in any period t, buy at the lowest quoted price.

Firm i, $i = 1, 2$:

I. Quote price $p_{it} = \nu$ as long as no one has deviated. If a deviation occurs at t, and if consumer buys from the defector, stay in I. If consumer does *not* buy from the defector, move to II.

II. Quote price $p_{i\tau} = c$ for all remaining periods.

Let us check whether these strategies constitute a subgame perfect equilibrium. Consider first the consumer.

1. If no one has deviated, and $p_t = (v, v)$, she can only accept the price and get a payoff of 0.

2. If a price cut is observed in period t, and the lowest quoted price is $p' = v - \epsilon$, consumer does not buy at p' if and only if

$$(1-\delta)\left[0 + (v-c)\sum_{t=1}^{\infty}\delta^t\right] \geqslant (1-\delta)\left[\epsilon + 0\right] \Leftrightarrow \epsilon \leqslant \frac{\delta(v-c)}{1-\delta} \equiv \epsilon^*. \quad (3)$$

Consider then firm i.

1. Suppose no one has deviated prior to time t. If firm continues quoting the monopoly price, it gets a payoff of $\frac{1}{2}(v-c)$. If it deviates and quotes a price $p_{it} = v - \epsilon$, what would its payoffs be? Given equation (3), the best deviation available for firm i is to quote a price $p_{it} = v - \epsilon^\star$,. At this price the consumer would just be willing to purchase, yielding the firm immediate gains as long as $v - \epsilon^\star - c > \frac{1}{2}(v-c) \Leftrightarrow \epsilon^\star \leqslant \frac{1}{2}(v-c)$. Deviation is not profitable if and only if

$$\frac{1}{2}(v-c) \geqslant (v-c) - \frac{\delta(v-c)}{1-\delta} \Leftrightarrow \delta \geqslant \frac{1}{3}.$$

2. Suppose that a deviation has occurred in $t-1$ (and the consumer bought from the defector). The choice that firms faces is the same as what it faces along the collusive path.

3. Suppose that a deviation has occurred in $t-1$ (and the consumer did not buy from the defector). This leads to stage NE.

Therefore, collusive equilibrium exists iff $\delta \geqslant \frac{1}{3}$, which is less than in (a).