**Problem 1** *Two firms simultaneously decide whether to enter a market. Firm $i$'s entry cost is $c_i \sim [0, \infty)$, and this is private information to firm $i$. Parameters $c_i$ are drawn independently from a distribution with a strictly positive density $f(\cdot)$. Firm $i$ has payoff $\Pi^m - c_i$ if $i$ is the only firm to enter and $\Pi^d - c_i$ if both firms enter. Not entering yields a payoff $0$. Assume that $\Pi^m > \Pi^d > 0$.*

*a)* Formulate the game as a Bayesian game.

*b)* Find the Bayesian Nash equilibrium of the game. Can you show that it is unique?

*c)* Analyze the game assuming that the firms make their entry decisions sequentially (say, firm 1 enters first and firm 2 decides about entry after observing firm 1's decision)

**Problem 2** *Two partners must dissolve their partnership. Partner 1 currently owns share $s$ of the partnership, partner 2 owns share $1 - s$. The partners agree to play the following game: Partner 1 names a price $p$, and partner 2 then chooses to buy 1's share for $ps$ or sell his share for $p(1 - s)$. Suppose it is common knowledge that the partners' valuations from owning the whole partnership are independently and uniformly distributed on $[0, 1]$, but each partner's valuation is private information. Formulate the game as a Bayesian game and find the perfect Bayesian equilibria.*

**Problem 3** *Consider private provision of public goods with incomplete information. Each player has a private cost $\theta_i \in [0, 2]$ of providing the public good. Suppose that costs are independent and uniformly distributed. The aim of this excercise is to find a symmetric perfect Bayesian equilibrium of a twice repeated version of the game. The payoffs per period are:*

| | *Contribute* | *Do not contribute* |
|---|---|---|
| *Contribute* | $1 - \theta_1, 1 - \theta_2$ | $1 - \theta_1, 1$ |
| *Do not contribute* | $1, 1 - \theta_2$ | $0, 0$ |

*a)* Assume first that the game is played only once, and the players choose simultaneously whether or not to contribute. Find the Bayesian equilibrium of the game.

*b)* Consider next the case where the game is repeated twice. The players first choose simultaneously whether or not to contribute in the first period. Then, after observing each others' actions, they choose simultaneously whether or not to contribute in the second period. Both players maximize the sum of payoffs over the two periods. Define the strategies in the game.

*c)* Argue that if there is a symmetric equilibrium strategy profile, then there must be some cutoff type $\widehat{\theta} \in (0,1)$ such that $i$ contributes in the first period if and only if $\theta_i \leqslant \widehat{\theta}$.

*d)* Suppose that $i$ contributes in the first period if and only if $\theta_i \leqslant \widehat{\theta}$, where $\widehat{\theta} \in (0,1)$, $i = 1,2$. Derive the posterior beliefs of the players in all information sets of the second period.

*e)* Solve the second-period equilibrium if neither player contributed in the first period.

*f)* Solve the second-period equilibrium if both players contributed in the first period.

*g)* Solve the second-period equilibrium if one player contributed and the other did not contribute in the first period.

*h)* Using the continuation payoffs for the second period derived above, solve for the cutoff $\widehat{\theta}$ such that a player with $\theta_i = \widehat{\theta}$ is indifferent between contributing and not contributing in the first period. Argue that you have derived a symmetric perfect Bayesian equilibrium of the game.

*i)* Is $\widehat{\theta}$ lower or higher than the corresponding equilibrium cutoff of the one-period version of the game? Discuss the intuition for this result.

**Problem 4** *Consider the following common values auction. There are two bidders whose types* $\theta_i$ *are independently drawn from a uniform distribution on* $[0, 100]$.

*The value of the object to both bidders is the sum of the types, i.e.* $\theta_i + \theta_j$. *The object is offered for sale in a first price auction. Hence the payoffs depend on the bids* $b_i$ *and types as follows (we ignore ties for convenience):*

$$u_i\left(b_i, b_j, \theta_i, \theta_j\right) = \begin{cases} \theta_i + \theta_j - b_i \text{ if } b_i > b_j, \\ 0 \text{ otherwise.} \end{cases}$$

*a)* Show by a direct computation that the linear strategies where $b_i = \theta_i$ for $i = 1, 2$ form a Bayesian equilibrium in this game.

*b)* If $\theta_i = 1$, the equilibrium bid is 1, but it might seem that the expected value of the object is 1+50=51. Why doesn't the bidder behave more aggressively?

*c)* Analyze the game above as a second price auction. Does the game have a dominant strategy equilibrium? Find a Bayesian Nash equilibrium of the game. (Hint: Think carefully about the event where changing one's own bid changes one's payoff. What does this imply about the bid of the other player? In symmetric equilibrium, what does this imply about the type of the other player? Alternatively, you may use the guess and verify method of the previous question and verify that a linear symmetric equilibrium exists.).

**Problem 5 (Global games)** *Two players choose between actions "Invest" and "Do not invest". Payoffs are as follows:*

| | *Invest* | *Do not invest* |
|---|---|---|
| *Invest* | $\theta, \theta$ | $\theta - 1, 0$ |
| *Do not invest* | $0, \theta - 1$ | $0, 0$ |

*a)* Find the Nash equilibria of the game for different values of $\theta$, when $\theta$ is common knowledge.

*b)* Suppose next that $\theta$ is not known to either of the players, but each player observes an independent private signal $x = \theta + \varepsilon_i$, where $\varepsilon_i$ is normally distributed with mean 0 and standard deviation $\sigma$. We assume here that the prior of $\theta$ is uniform on the whole real line. Such a uniform distribution over an infinitely long interval is called "improper". These distributional assumptions imply that the posterior of $\theta$ for a player that observes signal $x$ is a normal distribution with mean $x$ and standard deviation $\sigma$. What is the posterior of player $i$ who observed $x$ about the signal $x'$ of the other player?

*c)* Define a cut-off strategy in this game. Show that if player $-i$ is using an increasing cut-off strategy (so that investment is more likely for high signals), then the best response of $i$ is to use a cut-off strategy.

*d)* Find a Bayesian Nash equilibrium in cut-off strategies.

*e)* Show that the Bayesian Nash equilibrium that you derived above is the unique strategy profile surviving the iterated deletion of strictly dominated strategies.