Problem 1 *Two firms simultaneously decide whether to enter a market. Firm i's entry cost is $c_i \sim [0, \infty)$, and this is private information to firm i. Parameters c_i are drawn independently from a distribution with a strictly positive density $f(\cdot)$. Firm i has payoff $\Pi^m - c_i$ if i is the only firm to enter and $\Pi^d - c_i$ if both firms enter. Not entering yields a payoff 0. Assume that $\Pi^m > \Pi^d > 0$.*

a) Formulate the game as a Bayesian game.

Solution. The game is $\Gamma = \langle N, \{A_i\}, \{\Theta_i\}, \{p_i\}, \{s_i\}, \{u_i\}\rangle$, where
$N = \{1, 2\}$ is the set of players (the two firms),
$\{A_i\} = \{A_1, A_2\}$ is the collection of action sets where $A_i = \{\text{enter}, \text{not enter}\}$,
$\{\Theta_i\} = \{\Theta_1, \Theta_2\}$ is the collection of type sets (entry costs) where $\Theta_i = [0, \infty)$,
$\{p_i\} = \{p_1, p_2\}$ is the collection of beliefs on types where $p_i = f(\cdot)$,
$\{s_i\} = \{s_1, s_2\}$ is the set of strategies $s_i : \Theta_i \to A_i$; and
$\{u_i\} = \{u_1, u_2\}$ is the set of payoff functions such that

$$u_i\left(a_i, a_j; c_i, c_j\right) = \begin{cases} \Pi^m - c_i, & \text{if } a_i = \text{enter}, a_j = \text{not enter}; \\ \Pi^d - c_i, & \text{if } a_i = \text{enter}, a_j = \text{enter}; \\ 0, & \text{whenever } a_i = \text{not enter}. \end{cases}$$

We can write the payoff matrix of this game as

	Enter	Not enter
Enter	$\Pi^d - c_1, \Pi^d - c_2$	$\Pi^m - c_1, 0$
Not enter	$0, \Pi^m - c_2$	$0, 0$

b) Find the Bayesian Nash equilibrium of the game. Can you show that it is unique?

Solution. A Bayesian Nash equilibrium is a pair of strategies $(s_1(c_1), s_2(c_2))$ such that $s_i(c_i)\ E_{c_j}(u_i(a_i, a_j; c_i, c_j)) > E_{c_j}(u_i(a_i', a_j; c_i, c_j)$ for all $a_i' \in A_i$ for both firms 1 and 2.

The expected payoff from choosing $s_i(c_i) = enter$ is

$$E_{c_j}\left[u_i\left(enter, a_j; c_i, c_j\right)\right] = \Pr\left[s_j\left(c_j\right) = enter\right]\left(\Pi^d - c_i\right) + \Pr\left[s_j\left(c_j\right) = not\ enter\right]\left(\Pi^m - c_i\right),$$

which is a decreasing function of c_i.

The expected payoff from choosing $s_i(c_i) = not\ enter$ equals zero and is independent of c_i. Thus, there is a cutoff strategy $c^\star$ such that $s_i(c_i) = enter$ if $c_i \leqslant c^\star$ and $s_i(c_i) = not\ enter$ if $c_i > c^\star$. When $c_i = c^*$, firm i is indifferent between entering and not entering, that is

$$\begin{aligned}
\left(\Pi^d - c^*\right)\Pr(c_j \leqslant c^*) + (\Pi^m - c^*)\Pr(c_j > c^*) &= 0 \\
\Leftrightarrow \quad \left(\Pi^d - c^*\right)F(c^*) + (\Pi^m - c^*)\left[1 - F(c^*)\right] &= 0 \\
\Leftrightarrow \quad \Pi^d F(c^*) + \Pi^m\left[1 - F(c^*)\right] - c^* &= 0
\end{aligned}$$

$$\text{or} \qquad F(c^*) = \frac{\Pi^m - c^*}{\Pi^m - \Pi^d}, \tag{1}$$

where $F(\cdot)$ denotes the cumulative distribution function of $f(\cdot)$. The left-hand side of equation (1) is monotonically increasing, whereas the right-hand side of (1), denoted by $RHS(c)$, is monotonically decreasing in c. Furthermore, $F(0) = 0 < RHS(0) = \frac{\Pi^m}{\Pi^m - \Pi^d}$ and $F(\infty) = 1 > RHS(\infty) = -\infty$, and hence the left- and right-hand schedules must intersect exactly once. Thus, there is a symmetric equilibrium such that firm i enters if $c_i \leqslant c^*$ and does not enter otherwise. Further, the cutoff strategy c^* is defined by equation (1).

What it comes to uniqueness, we have shown that the solution above is unique *symmetric* equilibrium. Hence, what is left to solve is the possible existence of asymmetric equilibria with own cutoff points c_i^* for each $i \in \{1, 2\}$. Any asymmetric equilibrium with cutoff costs (c_1^*, c_2^*) must satisfy

$$F(c_1^*) = \frac{\Pi^m - c_2^*}{\Pi^m - \Pi^d} \qquad \text{and} \qquad F(c_2^*) = \frac{\Pi^m - c_1^*}{\Pi^m - \Pi^d}$$

$$\text{or} \quad c_i^* = \Pi^m - \left(\Pi^m - \Pi^d\right)F(\Pi^m - \left(\Pi^m - \Pi^d\right)F(c_i^*)), \tag{2}$$

for $i \in \{1, 2\}$. If there is a unique solution to (2), then the equilibrium is unique.

c) Analyze the game assuming that the firms make their entry decisions sequentially (say, firm 1 enters first and firm 2 decides about entry after observing firm 1's decision)

Solution. Let player 1 be the one that moves first. Let's try to find the equilibrium of the game by solving it by backward induction. There are two decision nodes for player 2. In the one in which player 1 has not entered, player 2 will enter if $c_2 \leqslant \pi^m$. In the one in which player 1 has entered: player 2 will enter if $c_2 \leqslant \pi^d$. So player 1 will enter if $c_1 \leqslant \Pr(c_2 \leqslant \pi^d)\pi^d + (1 - \Pr(c_2 \leqslant \pi^d)\pi^m = F(\pi^d)\pi^d + (1 - F(\pi^d))\pi^m$.

Problem 2 *Two partners must dissolve their partnership. Partner 1 currently owns share* s *of the partnership, partner 2 owns share* $1 - s$*. The partners agree to play the following game: Partner 1 names a price* p*, and partner 2 then chooses to buy 1's share for* ps *or sell his share for* $p\,(1-s)$*. Suppose it is common knowledge that the partners' valuations from owning the whole partnership are independently and uniformly distributed on* $[0, 1]$*, but each partner's valuation is private information. Formulate the game as a Bayesian game and find the perfect Bayesian equilibria.*

Solution. The game is $\Gamma = \langle N, \{A_i\}, \{\Theta_i\}, \{F_i\}, \{s_i\}, \{u_i\}\rangle$,
$N = \{1, 2\}$,
$\{A_i\} = \{A_1, A_2\}$, where $A_1 = [0, \infty)$ and $A_2 = \{buy, sell\}$,
$\{\Theta_i\} = \{\Theta_1, \Theta_2\}$, where $\Theta_i = [0, 1]$,
$\{F_i\} = \{F_1, F_2\}$, where F_i is the uniform distribution,
$\{s_i\} = \{s_1, s_2\}$, where $s_1 : \Theta_1 \to A_1$ and $s_2 : \Theta_2 \to A_2^{A_1}$, and
$\{u_i\} = \{u_1, u_2\}$, where

$$u_1(p, a_2; v_1, v_2) = \begin{cases} ps & \text{if } a_2 = buy, \\ (v_1 - p)(1-s) + v_1 s & \text{if } a_2 = sell. \end{cases}$$

and

$$u_2(p, a_2; v_1, v_2) = \begin{cases} (v_2 - p)\,s + v_2(1-s) & \text{if } a_2 = buy, \\ p\,(1-s) & \text{if } a_2 = sell. \end{cases}$$

To find the perfect Bayesian equilibria, suppose Partner 1 names a price p, Partner 2 buys (sells) if

$$v_2 > (<)\, p.$$

Thus,

$$a_2^*(p; v_2) = \begin{cases} \text{buy} & \text{if } v_2 > p, \\ \text{sell} & \text{if } v_2 < p. \end{cases}$$

Given that, Partner 1 chooses $p \in [0, \infty)$ to maximize

$$\begin{aligned} & \int_0^1 u_1(p, a_2^*(v_2); v_1, v_2)\, dF(v_2) \\ = & \int_0^p u_1(p, \text{sell}; v_1, v_2)\, dv_2 + \int_p^1 u_1(p, \text{buy}; v_1, v_2)\, dv_2 \\ = & \int_0^p ((v_1 - p)(1 - s) + v_1 s)\, dv_2 + \int_p^1 ps\, dv_2 \\ = & ((v_1 - p)(1 - s) + v_1 s)\, p + p(1 - p)\, s \end{aligned}$$

The first order condition is

$$-(1 - s)\, p + (v_1 - p)(1 - s) + v_1 s + (1 - 2p)\, s = 0.$$

Thus,

$$a_1^*(v_1) = \frac{v_1 + s}{2}.$$

Problem 3 *Consider private provision of public goods with incomplete information. Each player has a private cost $\theta_i \in [0, 2]$ of providing the public good. Suppose that costs are independent and uniformly distributed. The aim of this excercise is to find a symmetric perfect Bayesian equilibrium of a twice repeated version of the game. The payoffs per period are:*

	Contribute	*Do not contribute*
Contribute	$1 - \theta_1, 1 - \theta_2$	$1 - \theta_1, 1$
Do not contribute	$1, 1 - \theta_2$	$0, 0$

a) Assume first that the game is played only once, and the players choose simultaneously whether or not to contribute. Find the Bayesian equilibrium of the game.

Solution. Using similar reasoning as in problem 1, we already know that there should be a cutoff $\hat{\theta} < 1$ for which the players are indifferent between contributing and not given their prior information. Let's find this cutoff by writing payoffs for player j given that i uses this cutoff. Utility from contributing is $1 - \theta_j$. Utility from not contributing is $\Pr(\theta_i \leqslant \hat{\theta})$. Equating these and setting $\theta_j = \hat{\theta}$ gives:

$$1 - \hat{\theta} = \Pr(\theta_i \leqslant \hat{\theta}) = \frac{\hat{\theta}}{2} \iff \hat{\theta} = \frac{2}{3}.$$

b) Consider next the case where the game is repeated twice. The players first choose simultaneously whether or not to contribute in the first period. Then, after observing each others' actions, they choose simultaneously whether or not to contribute in the second period. Both players maximize the sum of payoffs over the two periods. Define the strategies in the game.

Solution. The possible outcomes in the first period are CC (both contributed), CD (1 contributed, 2 did not), DC (1 did not contribute, 2 did) and DD (neither contributed). The strategies need to define the probability of contributing for each $\theta_i \in [0, 2]$ for the first period and for the second period. We are going to use behavior strategies. Let's denote first period probability of contributing by $\sigma(\theta_i)$. That is $\sigma(\theta_i)$ is a mapping from $[0, 2]$ onto $[0, 1]$. The second period participation is conditional on the past play: $\sigma_i : \Theta_i \times \{C, D\}^2 \to [0, 1]$. These action probabilities are mappings from beliefs conditional on past play to $[0, 1]$.

c) Argue that if there is a symmetric equilibrium strategy profile, then there must be some cutoff type $\widehat{\theta} \in (0, 1)$ such that i contributes in the first period if and only if $\theta_i \leqslant \widehat{\theta}$.

Solution. Let's write down the expected utilities from contributing and not contributing. Let the probability that the other player contributes in the first period be ρ. Let $\rho^{CC}, \rho^{CD}, \rho^{DC}$ and ρ^{DD} be defined similarly as probabilities that the other player contributes in the second period given the history. Utility from contributing in the first period:

$$EU(C, \theta_i) = 1 - \theta_i + \rho \max\{1 - \theta_i, \rho^{CC}\} + (1 - \rho)\max\{1 - \theta_i, \rho^{CD}\}.$$

The first part of the expression is the first period's utility, while in the second period player i gets the higher of $1 - \theta_i$ or ρ^h, where ρ^h depends on whether the other player contributed in the first period or not.

Utility from not contributing is:

$$EU(D, \theta_i) = \rho + \rho max\{1 - \theta_i, \rho^{DC}\} + (1 - \rho)max\{1 - \theta_i, \rho^{DD}\}$$

Taking the difference of these two gives

$$EU(C, \theta_i) - EU(D, \theta_i) =$$
$$1 - \theta_i + \rho max\{1 - \theta_i, \rho^{CC}\} +$$
$$(1 - \rho)max\{1 - \theta_i, \rho^{CD}\} - \rho - \rho max\{1 - \theta_i, \rho^{DC}\} - (1 - \rho)max\{1 - \theta_i, \rho^{DD}\}.$$

To show that there is some cutoff type $\hat{\theta} \in (0, 1)$, we need to show that $EU(C, \theta_i) - EU(D, \theta_i)$ equals zero for some $\theta_i \in (0, 1)$ (i.e. for the cutoff type) and that it is decreasing in θ_i (i.e. types lower than $\hat{\theta}$ want to participate). The first part means that there is a type who is indifferent between contributing or not and the second part implies that types below that type will want to contribute and types above will not want to contribute.

Let's first show that $h(\theta_i) := EU(C, \theta_i) - EU(D, \theta_i)$ is decreasing in θ_i. Although,$h(\theta_i)$ is not differentiable everywhere (at most at four different points), but where it is its derivative with regards to θ_i is at most 0. To see this note that the derivative of the first term is always -1 and only the last two terms, $\rho max\{1 - \theta_i, \rho^{DC}\}$, $(1 - \rho)max\{1 - \theta_i, \rho^{DD}\}$ can have a positive sign. As these terms add up to at most to 1, the derivative is at most 0 ($p + (1 - p) = 1$). Thus, $EU(C, \theta_i) - EU(D, \theta_i)$ is at least weakly decreasing in θ_i.

To prove that $h(\theta_i)$ is zero for some θ_i, first note that $\rho < 1$. Since it is dominant for type $\theta > 1$ to not contribute in the second period and $\theta \sim unif[0, 2]$, the extra second period benefit for the type $\theta > 1$ who contributes in the first period is at most $-1/2$. As a result, type θ close to 2 would not contribute in the first period. Then note that $EU(C, \theta_i)$ must be positive for at least some $\theta_i = \epsilon > 0$ and equals at least $2 - 2\epsilon$, while $EU(D, \theta_i)$ is at most $\rho + 1$ (since ρ^{DC}, ρ^{DD} are at most 1). Thus there is an $\epsilon > 0$ such that $EU(C, \theta_i) > EU(D, \theta_i)$, namely when $2 - 2\epsilon > \rho + 1 \iff \epsilon < (1 - \rho)/2$. Thus all types $\theta_i \leqslant \epsilon$ will want to contribute with certainty. Furthermore, continuity of $h(\theta_i)$ and the facts that the utility is decreasing in θ_i and that $EU(C, 1) < EU(D, 1)$ imply that there must be a cutoff type $0 < \hat{\theta} < 1$ such that $h(\theta_i)$ is zero.

d) Suppose that i contributes in the first period if and only if $\theta_i \leqslant \widehat{\theta}$, where $\widehat{\theta} \in (0, 1)$, $i = 1, 2$. Derive the posterior beliefs of the players in all information sets of the second period.

Solution. There are four different information sets (histories) in the second period: CC, CD, DC and DD. If a player contributes in the first period then the other player knows that the player's type must be at or below $\hat{\theta}$, i.e. it is uniformly distributed on $[0, \hat{\theta}]$. If a player does not contribute then her value must be greater than $\hat{\theta}$, i.e it is uniformly distributed on $[\hat{\theta}, 2]$.

e) Solve the second-period equilibrium if neither player contributed in the first period.

Solution. First period outcome was DD, so $\theta_i \sim unif[\hat{\theta}, 2]$. The utility from contributing is, as previously, $1 - \theta_i$ and utility from not contributing equals $Pr(\theta_j \leqslant \hat{\theta}_{DD}) = (\hat{\theta}_{DD} - \hat{\theta})/(2 - \hat{\theta})$, where $\hat{\theta}_{DD}$ is the cutoff for the second period. To find the cutoff for the second period let's set these equal:

$$1 - \hat{\theta}_{DD} = \frac{\hat{\theta}_{DD} - \hat{\theta}}{2 - \hat{\theta}}.$$

Solving this yields $\hat{\theta}_{DD} = 2/(3 - \hat{\theta})$. So players contribute if $\theta_i < \hat{\theta}_{DD}$ and do not contribute otherwise.

f) Solve the second-period equilibrium if both players contributed in the first period.

Solution. Similar reasoning for CC (now $\theta_i \sim unif[0, \hat{\theta}]$) gives the following condition:

$$1 - \hat{\theta}_{CC} = Pr(\theta_j \leqslant \hat{\theta}_{CC}) = \frac{\hat{\theta}_{CC}}{\hat{\theta}}.$$

And solving this yields $\hat{\theta}_{CC} = \hat{\theta}/(1 + \hat{\theta})$.

g) Solve the second-period equilibrium if one player contributed and the other did not contribute in the first period.

Solution. Now the players know that the contributing player's type must be on $[0, \hat{\theta}]$ and the other player's on $[\hat{\theta}, 2]$. Let's say that player 1 contributed and player 2 did not. Since $\theta_1 < 1$, it is an equilibrium if player 1 contributes and player 2 does not contribute. Thus, there is an equilibrium such that the player who contributed in the first period contributes and the one that did not contribute in the first period does not contribute in the second period.

h) Using the continuation payoffs for the second period derived above, solve for the cutoff $\widehat{\theta}$ such that a player with $\theta_i = \widehat{\theta}$ is indifferent between contributing and not contributing in the first period. Argue that you have derived a symmetric perfect Bayesian equilibrium of the game.

Solution. We need to find the type who is indifferent between contributing and not contributing in the first period. Writing out the indifference condition:

$$\begin{gathered} EU(C, \theta_i) = EU(D, \theta_i) \iff \\ \Pr(\theta_j \leqslant \hat{\theta})v_2(C, C) + \Pr(\theta_j > \hat{\theta})v_2(C, D) = \Pr(\theta_j \leqslant \hat{\theta})v_2(D, C) + \Pr(\theta_j > \hat{\theta})v_2(D, D) \iff \\ \frac{\hat{\theta}}{2}v_2(C, C) + (1 - \frac{\hat{\theta}}{2})v_2(C, D) = \frac{\hat{\theta}}{2}v_2(D, C) + (1 - \frac{\hat{\theta}}{2})v_2(D, D). \end{gathered}$$

To simplify things, note that the cutoff type ($\theta_i = \hat{\theta}$) will contribute in the second period after CD and DD only, so we can write the continuation payoffs as

$$\begin{aligned} v_2(C, C) &= 1 - \hat{\theta} + \Pr(\theta_j \leqslant \hat{\theta}_{CC}) = 1 - \hat{\theta} + \frac{1}{1 + \hat{\theta}}. \\ v_2(C, D) &= 2(1 - \hat{\theta}). \\ v_2(D, C) &= 2. \\ v_2(D, D) &= 1 - \hat{\theta}. \end{aligned}$$

$v_2(C, C)$ follows from $\hat{\theta}_{CC} < \hat{\theta}$ and $v_2(D, D)$ follows from $\hat{\theta}_{DD} > \hat{\theta}$.

Plugging the continuation payoffs in:

$$\frac{\hat{\theta}}{2}(1-\hat{\theta}+\frac{1}{1+\hat{\theta}})+(1-\frac{\hat{\theta}}{2})(2(1-\hat{\theta}))=\frac{\hat{\theta}}{2}2+(1-\frac{\hat{\theta}}{2})(1-\hat{\theta}) \iff$$
$$\frac{\hat{\theta}}{2}(1-\hat{\theta}+\frac{1}{1+\hat{\theta}})+(1-\frac{\hat{\theta}}{2})(1-\hat{\theta})=\hat{\theta} \iff$$
$$(1-\hat{\theta})+\frac{\hat{\theta}}{2}(\frac{1}{1+\hat{\theta}})=\hat{\theta} \iff$$
$$4\hat{\theta}^2+\hat{\theta}-2=0.$$
$$\text{So, } \hat{\theta}=\frac{-1+\sqrt{33}}{8}<\frac{2}{3}.$$

i) Is $\widehat{\theta}$ lower or higher than the corresponding equilibrium cutoff of the one-period version of the game? Discuss the intuition for this result.

The cutoff is lower than in (a). In this equilibrium of the twice repeated game, there is a greater incentive to free ride in the first period, because a contributing player will also contribute in the second period with a high probability. Thus only low cost players will contribute in the first period.

Problem 4 *Consider the following common values auction. There are two bidders whose types* θ_i *are independently drawn from a uniform distribution on* $[0, 100]$. *The value of the object to both bidders is the sum of the types, i.e.* $\theta_i + \theta_j$. *The object is offered for sale in a first price auction. Hence the payoffs depend on the bids* b_i *and types as follows (we ignore ties for convenience):*

$$u_i\left(b_i, b_j, \theta_i, \theta_j\right) = \begin{cases} \theta_i + \theta_j - b_i \text{ if } b_i > b_j, \\ 0 \text{ otherwise.} \end{cases}$$

a) Show by a direct computation that the linear strategies where $b_i = \theta_i$ for $i = 1, 2$ form a Bayesian equilibrium in this game.

Solution. Let's calculate the best-reply of player i given that $j \neq i$ plays the linear strategy $b_j = \theta_j$. The expected utility of player i equals

$$\begin{aligned} Eu_i\left(b_i, b_j, \theta_i, \theta_j\right) &= \Pr\left(b_i > b_j\left(\theta_j\right)\right)\left(\theta_i + E\left(\theta_j | b_i > b_j\left(\theta_j\right)\right) - b_i\right) \\ &= F\left(b_i\right)\left(\theta_i + \tfrac{1}{F(b_i)}\textstyle\int_0^{b_i} \theta_j f\left(\theta_j\right) d\theta_j - b_i\right) \\ &= \tfrac{b_i}{100}\left(\theta_i + \tfrac{100}{b_i}\left(\textstyle\int_0^{b_i} \tfrac{\theta_j}{100} d\theta_j\right) - b_i\right) \\ &= \tfrac{b_i}{100}\left(\theta_i + \tfrac{100}{b_i}\left(\tfrac{b_i^2}{200}\right) - b_i\right) \\ &= \tfrac{b_i}{100}\left(\theta_i + \tfrac{b_i}{2} - b_i\right) = \tfrac{b_i}{100}\left(\theta_i - \tfrac{b_i}{2}\right). \end{aligned}$$

Taking the first order condition with regards to b_i yields

$$\frac{1}{100}\left(\theta_i - \frac{b_i}{2}\right) - \frac{b_i}{200} = 0 \Leftrightarrow b_i = \theta_i.$$

That is it is a best response to play $b_i = \theta_i$ against $b_j = \theta_j$.

b) If $\theta_i = 1$, the equilibrium bid is 1, but it might seem that the expected value of the object is 1+50=51. Why doesn't the bidder behave more aggressively?

Solution. The players only get the object if their bid is winning. They should consider only the expected value of the object conditional on their bid being the winning bid. In the equilibrium, the expected value of the object conditional on winning equals $\theta_i + E\left(\theta_j \mid b_i > b_j\right) = \theta_i + \frac{b_i}{2}$. So, if $\theta_i = 1$, the expected value conditional on winning is 1.5 although the unconditional expected value is 51.

c) Analyze the game above as a second price auction. Does the game have a dominant strategy equilibrium? Find a Bayesian Nash equilibrium of the game. (Hint: Think carefully about the event where changing one's own bid changes one's payoff. What does this imply about the bid of the other player? In symmetric equilibrium, what does this imply about the type of the other player? Alternatively, you may use the guess and verify method of the previous question and verify that a linear symmetric equilibrium exists.).

Solution. In a second price auction, the payoff structure is

$$u_i\left(b_i, b_j, \theta_i, \theta_j\right) = \begin{cases} \theta_i + \theta_j - b_j \text{ if } b_i > b_j, \\ 0 \text{ otherwise.} \end{cases}$$

Let's assume that the strategies are of the form $b_i\left(\theta_i\right) = a\theta_i$, where a is a constant. Player i's expected payoff from an arbitrary bid b_i when $b_j = a\theta_i$

$$\begin{aligned} Eu_i(b_i, b_j, \theta_i, \theta_j) &= \Pr(b_i > b_j)E(\theta_i + \theta_j - b_j \mid b_i > b_j) \\ &= (\theta_i + E(\theta_j - a\theta_j \mid b_i > a\theta_j))\tfrac{b_i}{100a} \\ &= (\theta_i + (1-a)\tfrac{b_i}{2a})\tfrac{b_i}{100a} = \tfrac{b_i\theta_i}{100a} + (1-a)\tfrac{b_i^2}{200a^2}. \end{aligned}$$

Taking the first order condition with regards to b_i yields:

$$\frac{\theta_i}{100a} + (1-a)\frac{2b_i}{200a^2} = 0 \iff b_i = \frac{a}{a-1}\theta_i.$$

And since $b_i = a\theta_i$ we have $a = 2$.

There is another way to argue this by noting that changing your bid only has an effect on payoff if you change from being a loser to being the winner or vice versa. The switch happens at $b_i = b_j$ which together with symmetry and monotonicity implies that $\theta_i = \theta_j$. Furthermore the players must be indifferent between winning and losing, i.e.

$$\theta_i + \theta_j - b_j = 0 \iff \theta_i + \theta_i - b_i = 0 \iff b_i = 2\theta_i.$$

Is there a dominant strategy equilibrium? No. This is because in a common value setting like this the best response of player i always depends on the strategy of player j. Think any other strategy for the other player, e.g. $b_j = \theta_j$, and best response will be something else than $b_i = 2\theta_i$.

Problem 5 (Global games) *Two players choose between actions "Invest" and "Do not invest". Payoffs are as follows:*

	Invest	*Do not invest*
Invest	θ, θ	$\theta - 1, 0$
Do not invest	$0, \theta - 1$	$0, 0$

a) Find the Nash equilibria of the game for different values of θ, when θ is common knowledge.

Solution. The equilibrium depends on the value of θ. If $\theta < 0$, (B,B) is the unique NE. If $\theta = 0$ (A,A) and (B,B) are both NE. If $0 < \theta \leqslant 1$ there are three equilibria: (A,A), (B,B) and a mixed strategy $(1-\theta, \theta)$ equilibrium. If $\theta > 1$, (A,A) is a unique NE.

b) Suppose next that θ is not known to either of the players, but each player observes an independent private signal $x = \theta + \varepsilon_i$, where ε_i is normally distributed with mean 0 and standard deviation σ. We assume here that the prior of θ is uniform on the whole real line. Such a uniform distribution over an infinitely long interval is called "improper". These distributional assumptions imply that the posterior of θ for a player that observes signal x is a normal distribution with mean x and standard deviation σ. What is the posterior of player i who observed x about the signal x' of the other player?

Solution. From player i's perspective, player j's signal is a sum of two normally distributed random variables. Hence, it is also a normally distributed random variable: $x_j | x_i \sim N(x_i, 2\sigma^2)$ (mean and variance are easy to calculate by using standard properties of E and Var).

c) Define a cut-off strategy in this game. Show that if player $-i$ is using an increasing cut-off strategy (so that investment is more likely for high signals), then the best response of i is to use a cut-off strategy.

Solution. Strategies are actions or mixtures of actions for each signal. Actions are here simply A and B. We thus can define pure strategies as a mapping from the set of signals to the set of actions: $\sigma_i : \mathbb{R} \to \{A, B\}$. An increasing cutoff strategy is of the form:

$$s_i = \begin{cases} A \text{ if } s_i \geqslant \hat{s_i} \\ B \text{ otherwise.} \end{cases}$$

Expected payoff of playing A instead of B is increasing if the other player is using an increasing strategy. Expectation of θ is increasing in the player's own signal and so is the probability that the other player plays A (player j's signal is increasing in the player i's signal and player j follows an increasing cutoff rule). Monotonicity implies that a best response is a cutoff strategy.

d) Find a Bayesian Nash equilibrium in cut-off strategies.

Solution. We already know that the best response for an increasing cutoff strategy is a cutoff strategy. The only task remaining is to find the cutoff:

$$Eu_i(A|\hat{x_i}) = Eu_i(B|\hat{x_i}) = 0 \iff E(\theta|\hat{x_i}) - Pr(x_j < \hat{x_j}|\hat{x_i}) = 0 \iff \hat{x_i} = F_{\hat{x_i}}(\hat{x_j}), \quad (3)$$

where $F_{\hat{x_i}}$ is a cdf for x_j conditional on $x_i = \hat{x_i}$. This implies that there is a symmetric equilibrium where $\hat{x_i} = \hat{x_j} = 1/2$.

e) Show that the Bayesian Nash equilibrium that you derived above is the unique strategy profile surviving the iterated deletion of strictly dominated strategies.

Solution. We assumed that the other player follows a cutoff strategy when finding the equilibrium in (c) and (d). Hence, the analysis above doesn't tell anything about the possibility of other equilibria. In addition, asymmetric cutoff could satisfy equation 3. In order to rule out other equilibria we proceed by iterated deletion of strictly dominated strategies.

Deletion round 1. Playing B when $x_i > 1$ is strictly dominated since $E(\theta|x_i) = x_i$.

Deletion round 2. First, define $\phi(k)$ such that $Pr(x_j \geqslant x_i + k|x_i) = 1/2 - \phi(k)$. Then, take $k_1(\sigma) > 0$ such that

$$1 - k_1(\sigma) - (1/2 + \phi(k_1(\sigma))) = 0. \tag{4}$$

This is possible because ϕ is continuous and $\phi(0) = 0$. Therefore, the LHS of equation 4 is continuous and strictly positive for small k_1. Playing B when $x_i > 1 - k_1(\sigma)$ is strictly dominated:

$$\begin{aligned} &Eu_i(A|x_i > 1 - k_1(\sigma)) = E(\theta|x_i > 1 - k_1(\sigma)) - Pr(\text{j plays B}|x_i > 1 - k_1(\sigma)) \\ &> 1 - k_1(\sigma) - (1/2 + \phi(k_1(\sigma))) = 0. \end{aligned}$$

The last inequality follows from the first deletion round: player j plays A always when $x_j > 1$.

Deletion round 3. Now player i knows that player j plays A when $x_j > 1 - k_1$. Therefore, $Pr(\text{j plays A}|x_i > 1 - k_1 - k_2) > Pr(x_j \geqslant 1 - k_1|x_i = 1 - k_1 - k_2) = 1/2 - \phi(k_2)$. We can use similar argumentation as above to delete playing B when $x_i > 1 - k_1(\sigma) - k_2(\sigma)$.

Deletion round n. We can proceed this way as long as we can find k_n satisfying:

$$1 - \sum_{i=1}^{n} k_i(\sigma) - (1/2 + \phi(k_n(\sigma))) = 0.$$

This is possible as long as $\sum_{i=1}^{n-1} k_i(\sigma) < 1/2$. Note that $\sum_{i=1}^{n} k_i = 1/2 - \phi(k_n)$. Hence, we will get arbitrarily close to $1/2$ when $k_n \to 0$, which happens when $n \to \infty$.

Now we have deleted all strategies that play B when $x_i > 1/2$. Similarly, one can delete strategies to play A when $x_i < 1/2$. Therefore, the symmetric equilibrium in (d) is a unique BNE.