# Supplementary material to "Uncertainty and energy saving investments"

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March 2010

**Abstract**

In this note we provide the explicit solution to the simple of model of Section 2. Proposition 1 follows from this solution. This proof builds on the myopia result explained in Section 3 of the paper. We derive the stopping rule for a myopic investor when the aggregate capacity $k$ is taken as given, and from this we derive the equilibrium path $k = \mathbf{k}(\hat{x})$ and its properties.

Define

$$\begin{aligned} \beta_1 &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} > 1, \\ \beta_2 &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} < 0. \end{aligned}$$

**Lemma 1** *Given the specification (1)-(4) in the main text, the optimal cut-off rule for a myopic investor as defined in Lemma 1 in the text is*

$$x^m(k) = \begin{cases} \frac{\delta\beta_1(B+C)}{rB(\beta_1-1)}\left(rI - \frac{AC}{B+C} + \frac{BC}{B+C}k\right) \text{ for } x \leq A - Bk \\ \left(-\frac{\beta_1(B+C)\left(\frac{A-Bk}{r} - I\right)}{(A-Bk)^{1-\beta_2} B\left(\frac{\beta_1}{r} + \frac{(1-\beta_1)}{\delta}\right)}\right)^{\frac{1}{\beta_2}} \text{ for } x > A - Bk. \end{cases} \tag{1}$$

**Proof.** Given $k$, the revenue process for an existing new plant is defined by

$$\begin{aligned} P(x;k) &= \begin{cases} \frac{C(A-Bk)}{B+C} + \frac{B}{B+C}x, \text{ for } x \leq A - Bk \\ A - Bk, \text{ for } x > A - Bk \end{cases} \\ &= \begin{cases} Q(k) + Rx, \text{ for } x \leq A - Bk \\ A - Bk, \text{ for } x > A - Bk \end{cases} \end{aligned}$$

where we use the definitions

$$Q(k) = \frac{C(A-Bk)}{B+C}, R = \frac{B}{B+C}.$$

The value of an existing plant, denoted by $V(x;k)$, satisfies the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 X^2 V''(x;k) + (r-\delta)\, xV'(x;k) - rV(x;k) + P(x;k) = 0,$$

where $r$ is the discount rate, and $\delta = r - \alpha$. The general solution of the equation is

$$\begin{aligned} V(x;k) &= \begin{cases} V_0(x;k), \text{ for } x \leq A - Bk \\ V_+(x;k), \text{ for } x > A - Bk \end{cases} \\ &= \begin{cases} B_1^0 x^{\beta_1} + B_2^0 x^{\beta_2} + \frac{Q(k)}{r} + \frac{Rx}{\delta}, \text{ for } x \leq A - Bk \\ B_1^+ x^{\beta_1} + B_2^+ x^{\beta_2} + \frac{A-Bk}{r}, \text{ for } x > A - Bk. \end{cases} \end{aligned}$$

where

The two boundary conditions $\lim_{x\to 0^+} V(x;k) = \frac{Q(k)}{r}$ and $\lim_{x\to\infty} V(x;k) = \frac{A-Bk}{r}$ imply that $B_2^0 = 0$ and $B_1^+ = 0$. The two remaining parameters would be easily solved by requiring that the first and second derivatives of the value functions match at $x = A - Bk$.

Denote the value of the option to install such a plant by $F(x;k)$. This must satisfy the following differential equation:

$$\frac{1}{2}\sigma^2 X^2 F''(x;k) + (r-\delta)\, XF'(x;k) - rF(x;k) = 0,$$

which has the general solution

$$F(x;k) = C_1 x^{\beta_1} + C_2 x^{\beta_2}.$$

The boundary condition $\lim_{x\to 0^+} F(x;k) = 0$ implies that $C_2 = 0$. The problem is to find $C_1$ and the myopic investment treshold $x^m$. There are two possible cases that must be considered separately: (1) $x^m \leq A - Bk$, and (2) $x^m > A - Bk$.

The boundary conditions in case $x^m \leq A - Bk$ are (taking into account that $B_2^0 = 0$):

$$\begin{aligned} C_1 x^{\beta_1} &= B_1^0 x^{\beta_1} + \frac{Q}{r} + \frac{Rx}{\delta} - I \\ \beta_1 C_1 x^{\beta_1 - 1} &= \beta_1 B_1^0 x^{\beta_1 - 1} + \frac{R}{\delta}. \end{aligned}$$

The ceiling $A - Bk$ is irrevelant in this case, and one can solve variable $C_1 - B_1^0$ instead of $C_1$. To see this, write these equations as

$$\begin{aligned} \left(C_1 - B_1^0\right) x^{\beta_1} &= \frac{Q(k)}{r} + \frac{Rx}{\delta} - I, \\ \beta_1 \left(C_1 - B_1^0\right) x^{\beta_1 - 1} &= \frac{R}{\delta}. \end{aligned}$$

From these, we obtain the following linear relationship between $x^m$ and $k$:

$$x^m = \frac{-\delta\beta_1\left(\frac{Q(k)}{r} - I_r\right)}{R\left(\beta_1 - 1\right)} = \frac{\delta\beta_1(B+C)}{rB(\beta_1 - 1)}(rI - \frac{AC}{B+C} + \frac{BC}{B+C}k). \tag{2}$$

The boundary conditions in case $x^m > A - Bk$ are

$$\begin{aligned} C_1 x^{\beta_1} &= B_2^+ x^{\beta_2} + \frac{A - Bk}{r} - I \\ \beta_1 C_1 x^{\beta_1 - 1} &= \beta_2 B_2^+ x^{\beta_2 - 1}. \end{aligned}$$

This implies that the investment trigger is given by the non-linear equation:

$$x^m = \left( - \frac{\beta_1\left(B+C\right)\left(\frac{A-Bk}{r} - I\right)}{(A-Bk)^{1-\beta_2} B \left(\frac{\beta_1}{r} + \frac{(1-\beta_1)}{\delta}\right)} \right)^{\frac{1}{\beta_2}}.$$

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For the properties of the equilibrium it is enough to focus on the case $x^m \leq A - Bk$. Let us now use the notation $\hat{x}$ for the equilibrium investment trigger which is defined by the myopic trigger $x^m(k)$. We can see from (1) that for $x^m \leq A - Bk$, the myopic investment trigger $x^m(k)$ defines the equilibrium capacity as a linear function of the current record $\hat{x}$

$$\mathbf{k}(\hat{x}) = \frac{r(\beta_1 - 1)}{\beta_1 \delta C}\hat{x} + \frac{AC - rI(B+C)}{BC}.$$

Let us now explain the role of volatility for the equilibrium description to apply. Recall that $\hat{x}^*$ is the equilibrium investment trigger at which $\hat{x}^* = x^m = P = A - Bk^*$. Using the formula for $x^m\,(k)$ as given in (2), we can solve $k^*$ from

$$\frac{\delta\beta_1(B+C)}{rB(\beta_1 - 1)}(rI - \frac{AC}{B+C} + \frac{BC}{B+C}k^*) = A - Bk^*, \tag{3}$$

which gives

$$\begin{aligned} k^* &= \frac{\beta_1(\delta AC + rAB - \delta rI(B+C)) - rAB}{B(\beta_1(rB + \delta C) - rB)}, \\ \hat{x}^* &= \frac{rI\delta\beta_1(B+C)}{\beta_1\delta C + rB(\beta_1 - 1)} \end{aligned}$$

where the latter equation is obtained by evaluating $x^m(k)$ at $k^*$. Consider now $k = 0$ and the condition (3). The ratio $\beta_1/(1-\beta_1)$ increases in $\sigma$ monotonically so that the left-hand side of (3) exceeds the right-hand side even at $k = 0$. This would imply that the market must shut down before new entry can take place. There is therefore a unique $\sigma^*$ such that equation (3) holds as equality when $k = 0$. For all $\sigma < \sigma^*$ we can find a strictly positive value for $k^*$ and thus for $\hat{x}^*$.

The investment trigger in terms of output price is

$$P_H(\hat{x}) = \frac{C(A - Bk)}{B + C} + \frac{B}{B + C}\hat{x} = rI + \frac{\beta_1 B(\delta - r) + rB}{\beta_1 \delta(B + C)}\hat{x} \text{ for } x \leq \hat{x}^*.$$

We see that the price is increasing in $\hat{x}$, implying contraction of output for $x \leq \hat{x}^*$. The price trigger is

$$P_H(\hat{x}) = A - B\mathbf{k}(\hat{x}) \text{ for } x > \hat{x}^*,$$

which is decreasing in $\hat{x}$. The output thus expands for $x > \hat{x}^*$.

The peak price follows by direct substitution

$$P_H(\hat{x}^*) = \frac{\beta_1 \delta r I(B + C)}{\beta_1(rB + \delta C) - rB},$$

which is increasing in $\sigma$. When $C \to 0$, the myopic investment trigger approaches

$$x^m \to \frac{\delta \beta_1 B}{rB(\beta_1 - 1)} rI,$$

which is independent of $k$. Thus, once this trigger is reached, there is a discrete one-time jump in the capacity path. This completes the proof of the Proposition 1.